

Available online at www.sciencedirect.com

ScienceDirect

Journal of Algebra 313 (2007) 61–99

**JOURNAL OF
Algebra**

www.elsevier.com/locate/jalgebra

The total coordinate ring of a wonderful variety

Michel Brion

*Université de Grenoble I, Département de Mathématiques, Institut Fourier, UMR 5582 du CNRS,
38402 Saint-Martin d'Hères Cedex, France*

Received 27 June 2006

Available online 24 January 2007

Communicated by Victor Kac, Ruth Kellerhals, Friedrich Knop, Peter Littelmann and Dmitri Panyushev

To Ernest Borisovich Vinberg on the occasion of his seventieth birthday

Abstract

We study the cone of effective divisors and the total coordinate ring of wonderful varieties, with applications to their automorphism group. We show that the total coordinate ring of any spherical variety is obtained from that of the associated wonderful variety by a base change of invariant subrings.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Spherical variety; Wonderful variety; Total coordinate ring

Contents

| | | |
|------|--|----|
| 1. | Introduction | 62 |
| 2. | Divisors on wonderful varieties | 64 |
| 2.1. | Wonderful varieties | 64 |
| 2.2. | Picard group | 67 |
| 2.3. | The cone of effective divisors | 70 |
| 2.4. | The connected automorphism group | 72 |
| 3. | The total coordinate ring of a wonderful variety | 76 |
| 3.1. | Definitions and basic properties | 76 |
| 3.2. | An algebraic description | 80 |
| 3.3. | Generators and relations | 84 |
| 4. | The total coordinate ring of a spherical variety | 86 |
| 4.1. | The equivariant divisor class group | 86 |

E-mail address: michel.brion@ujf-grenoble.fr.

0021-8693/\$ – see front matter © 2007 Elsevier Inc. All rights reserved.

doi:10.1016/j.jalgebra.2006.12.022

| | | |
|------------|---|----|
| 4.2. | The total coordinate ring | 90 |
| 4.3. | Structure | 92 |
| 4.4. | The automorphism group of a complete toroidal variety | 95 |
| References | | 98 |

1. Introduction

Wonderful varieties are smooth projective algebraic varieties where a semi-simple algebraic group G acts with finitely many orbits; one assumes, in addition, that the G -orbit closures look like the coordinate subspaces in an affine space \mathbb{A}^r . Examples include the flag varieties G/P (the case where $r = 0$), and the complete symmetric varieties of De Concini and Procesi (see [15]; here r is the rank of the symmetric space embedded in the variety as the open G -orbit).

The class of wonderful varieties forms a natural generalization of that of flag varieties; it also plays a central role in the theory of spherical varieties, as shown by Luna. Indeed, by [26], every wonderful variety is spherical, i.e., it contains an open orbit of a Borel subgroup of G . Conversely, the classification of spherical varieties may be reduced to that of wonderful varieties, see [28]. And the latter is known in many cases: when the “rank” r is at most 2 (see [2,20] for $r = 1$, and [40] for $r = 2$), and when the group G is of type A (see [28]) or D (see [7]). However, the problem of classifying wonderful varieties is still open in general, due to the complexity of their combinatorial invariants: the “spherical systems” of [28].

An important feature of this problem is that a given variety may be wonderful with respect to several group actions, which may also have the same orbits. For example, the projective space \mathbb{P}^n is of course a wonderful variety under SL_{n+1} , and (if n is odd) under Sp_{n+1} , both acting transitively. Moreover, \mathbb{P}^n is a wonderful variety under SO_{n+1} acting with two orbits: a smooth quadric hypersurface and its complement. In addition, \mathbb{P}^6 is a wonderful variety under the subgroup $G_2 \subset SO_7$ (acting with the same orbits), and \mathbb{P}^7 is a wonderful variety under $Spin_7$ (acting via the projectivization of its spin representation; the orbits are still a quadric and its complement). One may show that there is no other structure of a wonderful variety on a projective space. Another example is $\mathbb{P}^n \times \mathbb{P}^n$ blown up along the diagonal, which is a wonderful variety of rank 1 under the natural action of SL_{n+1} and also (if n is odd) a wonderful variety of rank 2 under Sp_{n+1} .

In the present paper, we study three closely related algebro-geometric invariants of wonderful varieties: the cone of effective divisors, the connected automorphism group, and the total coordinate ring. This yields insight into their combinatorial invariants, and into the classification of all wonderful group actions on a given variety.

To describe our results, recall that any wonderful variety X comes with a finite set of prime divisors called “colors,” which yields a basis of the Picard group. Thus, we may define (after [5, 14,18]) the total coordinate ring $R(X)$ as the direct sum of spaces of sections of all linear combinations of colors with integral coefficients. This ring carries a natural grading by the monoid of classes of effective divisors, $\text{Eff}(X)$.

It is easy to show that the algebra $R(X)$ and the monoid $\text{Eff}(X)$ are finitely generated when X is any spherical variety. For wonderful varieties, we obtain much more precise results. Namely, we first describe all the extremal rays of the cone $\text{Eff}(X)_{\mathbb{R}}$ of effective divisors (Theorems 2.2.3 and 2.4.1). They fall into two types: the pull-backs of Schubert divisors under fibrations $X \rightarrow X'$ where X' is a flag variety with Picard group \mathbb{Z} , and those “boundary divisors” (the G -orbit

closures of codimension 1) that are “fixed,” that is, any positive multiple has a unique section up to scalars.

The fixed divisors also appear when studying automorphisms of the wonderful G -variety X . Indeed, these divisors are all stable under the connected automorphism group $\mathrm{Aut}^0(X)$. We show that $\mathrm{Aut}^0(X)$ is semi-simple, as well as any closed connected subgroup G' containing the image of G ; moreover, X is still wonderful under the action of G' . In particular, X has a canonical structure of a wonderful variety under $\mathrm{Aut}^0(X)$; then its colors are the same as those associated with the G -action, and its boundary divisors are exactly the fixed divisors (see Theorem 2.4.2 for a more general statement).

In loose words, the colors and the fixed boundary divisors are intrinsic invariants of the variety X . Hence the same holds for a large part of Luna’s combinatorial invariants, by their definition in [28].

We then turn to the total coordinate ring $R(X)$, also known as the section ring, or Cox ring, or factorial hull. The canonical sections s_1, \dots, s_r of the boundary divisors are easily seen to form a regular sequence in $R(X)$ and to generate the invariant subring $R(X)^G$; moreover, all fibers of the corresponding flat morphism $q : \mathrm{Spec} R(X) \rightarrow \mathbb{A}^r$ are normal varieties (Proposition 3.1.1).

We also obtain an algebraic descriptions of $R(X)$ as a certain “Rees algebra” (Theorem 3.2.3), and a presentation by generators and relations. It should be emphasized that the simple description of the total coordinate ring of a toric variety (a polynomial ring over the coordinate ring of the acting torus, see [1,14]) does not extend in a straightforward way to a wonderful variety X . Specifically, the ring $R(X)$ still admits simple generators, namely, s_1, \dots, s_r and the G -submodules spanned by the canonical sections of colors, but their relations turn out to be rather complicated.

For example, $R(G/P)$ is the coordinate ring of the quasi-affine homogeneous space $G/[P, P]$, which is defined by quadratic equations. In the general case, the quotient $R(X)/(s_1, \dots, s_r)$ is also defined by quadratic equations, that one has to lift to $R(X)$ (Proposition 3.3.1). This yields some information on the (still largely open) problem of describing the product of any two simple G -submodules of $R(X)$; see Proposition 3.3.2. For certain complete symmetric varieties, more precise results are due to Chirivì, Littelmann and Maffei, see [12].

Finally, we study the total coordinate ring of an arbitrary spherical variety X under a connected reductive group G . Here X may be quite singular, so that it is natural to define the ring $R(X)$ via Weil divisors as in [5]; actually, one should consider G -linearized divisorial sheaves to keep track of the G -action. Then the Picard group is replaced with the equivariant divisor class group $\mathrm{Cl}^G(X)$, which has no canonical basis in general; in fact, this group may have torsion. So the definition of $R(X)$ requires some work, see 4.2 for details.

Once this is done, the ring $R(X)$ turns out to be obtained from the total coordinate ring of a wonderful variety by a base change. Specifically, the spherical G -variety X is equipped with an equivariant rational map $\varphi : X \dashrightarrow \mathbf{X}$, where \mathbf{X} is a wonderful variety under a semi-simple quotient \mathbf{G} of the group G ; see [28]. This yields an equivariant homomorphism $\varphi^* : R(\mathbf{X}) \rightarrow R(X)$; we show that the induced map $R(\mathbf{X}) \otimes_{R(\mathbf{X})^{\mathbf{G}}} R(\mathbf{X})^{\mathbf{G}} \rightarrow R(X)$ is an isomorphism (Theorem 4.3.2). This may be regarded as a universal property of the morphism $q : \mathrm{Spec} R(\mathbf{X})^{\mathbf{G}} \rightarrow \mathbb{A}^r$, a flat family of affine \mathbf{G} -varieties. The corresponding morphism to the “invariant Hilbert scheme” of [4] calls for investigation.

We also study the connected automorphism group $\mathrm{Aut}^0(X)$, where X is an arbitrary complete toroidal variety (i.e., the above rational map φ is a morphism). In contrast with the wonderful case, $\mathrm{Aut}^0(X)$ is generally far from being semi-simple (e.g., it is an n -dimensional torus when X is the projective n -space blown up at $n + 1$ general points) or even reductive (e.g., for Hirze-

bruch surfaces). However, we show that the “double centralizer” of G (that is, the centralizer in $\text{Aut}^0(X)$ of the equivariant connected automorphism group $\text{Aut}_G^0(X)$) is reductive, as well as any subgroup containing the image of G . Also, the subgroup of $\text{Aut}^0(X)$ preserving the boundary is reductive and depends only on the open G -orbit in X . For this, we relate these subgroups to the corresponding subgroups of $\text{Aut}^0(\mathbf{X})$, see Theorem 4.4.1.

For a complete symmetric variety X , the ring $R(X)$ has been considered by Chirivì and Maffei [13] who constructed a “standard monomial theory” in this setting. This ring also appears, somewhat implicitly, in Vinberg’s work [39] on reductive algebraic semigroups which has been the starting point of the present article. Indeed, as shown by Rittatore in [36,37], Vinberg’s “enveloping semigroup” $\text{Env}(G)$ is the affine variety with coordinate ring $R(\bar{G}_{\text{ad}})$, where \bar{G}_{ad} denotes the wonderful compactification of the adjoint group G_{ad} (a special case of a wonderful symmetric variety); see [33] and its references for related results. Throughout this paper, we shall specialize our results to the group compactifications \bar{G}_{ad} in a series of examples.

Notation. We shall use [19] as a general reference for algebraic geometry, and [30] for algebraic transformation groups.

The ground field k is algebraically closed, of characteristic zero. By a variety, we mean an integral separated scheme of finite type over k . We denote by $k(X)$ the field of rational functions on a variety X , and by $k[X] = \Gamma(X, \mathcal{O}_X)$ its algebra of global regular functions.

We denote by G a connected reductive linear algebraic group; we choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let $W := N_G(T)/T$ be the Weyl group, and U the unipotent part of B . The group of multiplicative characters $\mathcal{X}(T)$ is also denoted by Λ , and identified with the character group $\mathcal{X}(B)$. Let $\Phi \subset \Lambda$ be the root system of (G, T) with its subset of positive roots Φ^+ consisting of roots of (B, T) . This defines the subset $\Lambda^+ \subset \Lambda$ of dominant weights, which is also the set of highest weights of simple rational G -modules. For any $\lambda \in \Lambda$, we denote by $V(\lambda)$ the corresponding simple module.

2. Divisors on wonderful varieties

In this section, G is assumed to be semi-simple and simply-connected.

2.1. Wonderful varieties

Recall from [26] that a G -variety X is called *wonderful of rank r* if it satisfies the following conditions:

- (1) X is complete and non-singular.
- (2) X contains an open G -orbit X^0 , and the complement $X \setminus X^0$ (the *boundary*) is a union of r non-singular prime divisors X_1, \dots, X_r (the *boundary divisors*) with normal crossings.
- (3) The G -orbit closures in X are exactly the partial intersections $\bigcap_{i \in I} X_i$, where I is a subset of $\{1, \dots, r\}$.

In particular, X contains a unique closed G -orbit, the intersection of all boundary divisors; it follows that X is projective. We now recall a number of notions and results concerning wonderful varieties; see [28] and its references for details (some of our notation differs from [28]).

Denote by X_B^0 the open B -orbit in the wonderful variety X . Then X_B^0 is affine and contained in X^0 , so that $X^0 \setminus X_B^0$ is a union of finitely many prime B -stable divisors, called *colors*. Let \mathcal{D}

be the set of colors, regarded as divisors in X by taking closures; these are the prime B -stable divisors in X which are not G -stable. Moreover,

$$X \setminus X_B^0 = X_1 \cup \cdots \cup X_r \cup \bigcup_{D \in \mathcal{D}} D. \quad (2.1.1)$$

Let also

$$X_B := X \setminus \bigcup_{D \in \mathcal{D}} D,$$

this is an open B -stable subset of X , called the *big cell*. In fact, X_B is isomorphic to an affine space, and meets each G -orbit along its open B -orbit.

Each intersection $X_i \cap X_B$ has an equation $f_i \in k[X_B]$, uniquely defined up to a non-zero scalar. In particular, f_i is a B -eigenvector; let $\gamma_i \in \Lambda$ be the opposite of its weight. Then $\gamma_1, \dots, \gamma_r$ are the *spherical roots* of X . In fact, the spherical roots form a basis of a reduced root system Φ_X , and the corresponding Weyl group W_X is a subgroup of W . Moreover, each spherical root is either a positive root of Φ , or a sum of two such roots.

We may regard f_1, \dots, f_r as B -eigenvectors in $k(X)$; in fact, the multiplicative group of B -eigenvectors $k(X)^{(B)}$ is freely generated by f_1, \dots, f_r over its subgroup k^* of constants. Moreover, associating with each eigenvector its weight yields an injective homomorphism $k(X)^{(B)}/k^* \rightarrow \Lambda$; its image is the *weight group* $\Lambda(X)$, freely generated by the spherical roots.

Choose a base point $x \in X_B^0$ and denote by $H = G_x$ its isotropy group. This yields isomorphisms $X^0 \cong G/H$, $X_B^0 \cong BH/H \cong B/B \cap H$, and $\Lambda(X) \cong \mathcal{X}(B)^{B \cap H}$ (the group of characters of B which are invariant under $B \cap H$). We may choose T so that $T \cap H$ is a maximal diagonalizable subgroup of $B \cap H$; then $\mathcal{X}(B)^{B \cap H} \cong \mathcal{X}(T/T \cap H)$.

Also, recall that H has finite index in its normalizer $N_G(H)$; moreover, the equivariant automorphism group $\text{Aut}_G(X)$ is isomorphic to $\text{Aut}_G(G/H) \cong N_G(H)/H$ via restriction. In particular, $\text{Aut}_G(X)$ is finite.

Let $\pi : G \rightarrow G/H$ denote the projection. Each $D \in \mathcal{D}$ yields a $B \times H$ -stable divisor $\pi^{-1}(D)$ on G , which has an equation $f_D \in k[G]$ uniquely defined up to a non-zero scalar (since $k[G]$ is a UFD and $k[G]^* = k^*$). Thus, f_D is an eigenvector of $B \times H$, and $f_D(1) \neq 0$ so that f_D is uniquely determined by requiring that $f_D(1) = 1$. Likewise, each f_i is uniquely determined by requiring that $f_i(x) = 1$.

By [28, Lemma 6.2.2], the $B \times H$ -eigenvectors in $k(G)$ are exactly the Laurent monomials in the f_D , $D \in \mathcal{D}$; it follows that the $B \times H$ -eigenvectors in $k[G]$ are the monomials in the f_D 's. Let $(\omega_D, \chi_D) \in \mathcal{X}(B) \times \mathcal{X}(H)$ be the weight of f_D . Note that the restrictions to $B \cap H$ of ω_D and χ_D coincide. In other words, (ω_D, χ_D) is in the fibered product $\mathcal{X}(B) \times_{\mathcal{X}(B \cap H)} \mathcal{X}(H)$. In fact, we have the following useful result (probably known, but for which we could not supply a reference).

Lemma 2.1.1. *The abelian group $\mathcal{X}(B) \times_{\mathcal{X}(B \cap H)} \mathcal{X}(H)$ is freely generated by the pairs (ω_D, χ_D) , $D \in \mathcal{D}$. Moreover, $\mathcal{X}(H)$ is generated by the χ_D , $D \in \mathcal{D}$.*

Proof. Consider the action of the group $B \times H$ on G via $(b, h) \cdot g = bgh^{-1}$. The product BH is the orbit of the identity element 1; it is open in G (since $B/B \cap H$ is open in G/H), and the isotropy group of 1 equals $\text{diag}(B \cap H)$. It follows that each $B \times H$ -eigenvector in $k(G) =$

$k(BH) = k((B \times H)/\text{diag}(B \cap H))$ is determined up to a scalar by its pair of weights. Moreover, the set of these weights is $\mathcal{X}(B) \times_{\mathcal{X}(B \cap H)} \mathcal{X}(H)$. This implies the first assertion. For the second one, observe that the projection $\mathcal{X}(B) \times_{\mathcal{X}(B \cap H)} \mathcal{X}(H) \rightarrow \mathcal{X}(H)$ is surjective, since any character of $B \cap H$ extends to a character of B . \square

Any prime divisor D in X defines a discrete valuation v_D of the function field $k(X)$; its restriction to $k(X)^{(B)}$ yields a homomorphism

$$\rho(v_D): k(X)^{(B)}/k^* \cong \Lambda(X) \rightarrow \mathbb{Z}.$$

Let $v_i := \rho(v_{X_i})$ for $i = 1, \dots, r$, then $v_i(f_j) = \delta_{ij}$. In other words, v_1, \dots, v_r form a basis of the abelian group $\text{Hom}(\Lambda(X), \mathbb{Z})$, dual to the basis $(-\gamma_1, \dots, -\gamma_r)$. In particular, $\rho(v_D) = \sum_{i=1}^r \langle D, \gamma_i \rangle v_i$, where

$$\langle D, \gamma_i \rangle := -\rho(v_D)(f_i). \quad (2.1.2)$$

Also, $f_i = \prod_{D \in \mathcal{D}} f_D^{-\langle D, \gamma_i \rangle}$. Equivalently,

$$\sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle \omega_D = \gamma_i \quad \text{and} \quad \sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle \chi_D = 0. \quad (2.1.3)$$

The convex cone

$$\mathcal{V} := \text{Cone}(v_1, \dots, v_r) \subset \text{Hom}(\Lambda(X), \mathbb{R})$$

may be identified with the set of G -invariant discrete valuations of $k(X) = k(G/H)$. Thus, \mathcal{V} is called the *valuation cone* of the spherical homogeneous space G/H , and $\Lambda(X) = \Lambda(G/H)$ is called its *weight group*.

The cone \mathcal{V} and the map $\rho: \mathcal{D} \rightarrow \text{Hom}(\Lambda(G/H), \mathbb{R})$ are the main ingredients of the Luna–Vust classification of embeddings of G/H and, more generally, of any spherical homogeneous space. We refer to [21] for an exposition of this theory; we shall freely use some of its results in the sequel. We shall also need the following lemma.

Lemma 2.1.2. *The opposite of the valuation cone is contained in the cone generated by the $\rho(v_D)$, $D \in \mathcal{D}$.*

Proof. It suffices to show the inclusion of dual cones in $\Lambda(X)_{\mathbb{R}}$:

$$\text{Cone}(\rho(v_D), D \in \mathcal{D})^{\vee} \subseteq -\mathcal{V}^{\vee} = \text{Cone}(\gamma_1, \dots, \gamma_r).$$

Let $f \in k(X)^{(B)}$ such that $\rho(v_D)(f) \geq 0$ for all $D \in \mathcal{D}$. Then f is a regular function on G/H , and hence its weight λ is dominant. Thus, $(\lambda, \gamma_i) \geq 0$ for $i = 1, \dots, r$, where (\cdot, \cdot) denotes the scalar product on $\Lambda(X)_{\mathbb{R}} \subseteq \Lambda_{\mathbb{R}}$ defined by the Killing form of G . Since (\cdot, \cdot) is invariant under the Weyl group W_X of the root system Φ_X , it follows that λ is also dominant relatively to this root system and its basis of spherical roots. Thus, the coordinates of λ in this basis are non-negative, as follows from the equality $|W_X|\lambda = \sum_{w \in W_X} (\lambda - w(\lambda))$. \square

Example 2.1.3. Let Z denote the center of G , and $G_{\text{ad}} := G/Z$ the corresponding adjoint group. Then G_{ad} is a homogeneous space under $G \times G$ acting by left and right multiplication; the isotropy group of the base point 1 is $(Z \times Z) \text{diag}(G)$. By [15], G_{ad} admits a wonderful compactification $X = \bar{G}_{\text{ad}}$, of rank $r = \text{rk}(G)$. Its spherical roots (relatively to the Borel subgroup $B^- \times B$, where B^- denotes the unique Borel subgroup such that $B^- \cap B = T$) are the pairs $(-\alpha_i, \alpha_i)$ where $\alpha_1, \dots, \alpha_r$ are the simple roots of Φ . Thus, $\Lambda(X)$ may be identified with the root lattice of Φ ; then $\text{Hom}(\Lambda(X), \mathbb{Z})$ is the weight lattice, and v_1, \dots, v_r are the opposites of the fundamental co-weights. The valuation cone is the negative Weyl chamber. The map $\rho: \mathcal{D} \rightarrow \text{Hom}(\Lambda(X), \mathbb{Z})$ is injective, and its image consists of the simple co-roots $\alpha_1^\vee, \dots, \alpha_r^\vee$. Hence the integers $\langle D, \gamma_i \rangle$ are the coefficients of the Cartan matrix of the dual root system Φ^\vee .

2.2. Picard group

Since the wonderful variety X is non-singular, we may identify the Picard group $\text{Pic}(X)$ with the divisor class group. And since the complement of the union of the colors is an affine space, the classes of these colors form a basis of $\text{Pic}(X)$. We shall identify $\text{Pic}(X)$ with the free abelian group $\mathbb{Z}^{\mathcal{D}}$ and denote its elements by $\sum_{D \in \mathcal{D}} n_D [D]$, where $(n_D) \in \mathbb{Z}^{\mathcal{D}}$.

Recall from [10, Section 2.6] that the divisor $\sum_{D \in \mathcal{D}} n_D D$ is base-point-free (respectively ample) if and only if $n_D \geq 0$ (respectively > 0) for all $D \in \mathcal{D}$. In other words, the monoid $\text{Nef}(X)$ consisting of the classes of numerically effective (nef) divisors is generated by the classes of colors, and each nef divisor is base-point-free. In particular, the nef cone $\text{Nef}(X)_{\mathbb{R}}$ is simplicial.

Since G is semi-simple and simply-connected, each invertible sheaf on X admits a unique G -linearization, see [25, Section 2.4]. Thus, we shall also identify $\text{Pic}(X)$ with the group of isomorphism classes of G -linearized invertible sheaves, $\text{Pic}^G(X)$. We shall use the notation $\mathcal{O}_X(\sum_{D \in \mathcal{D}} n_D D)$ when considering an element of $\text{Pic}(X)$ as a G -linearized invertible sheaf. Each space of global sections $\Gamma(X, \mathcal{O}_X(\sum_{D \in \mathcal{D}} n_D D))$ has a canonical G -module structure.

In particular, each effective divisor admits a B -semi-invariant global section. In other words, the monoid $\text{Eff}(X)$ consisting of the classes of effective divisors is generated by the classes of colors and of boundary divisors. To express the latter in terms of the former, note that

$$\text{div}(f_i) = X_i - \sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle D \quad (2.2.1)$$

by (2.1.2), so that

$$[X_i] = \sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle [D] \quad \text{in } \text{Pic}(X). \quad (2.2.2)$$

We now describe the restriction map $i^*: \text{Pic}(X) \rightarrow \text{Pic}(G/H)$, where $i: G/H \rightarrow X$ denotes the inclusion. Recall the isomorphisms

$$\text{Pic}(G/H) \cong \text{Pic}^G(G/H) \cong \mathcal{X}(H),$$

where the latter is obtained by restriction to the base point. The inverse isomorphism is given by

$$\mathcal{X}(H) \rightarrow \text{Pic}^G(G/H), \quad \chi \mapsto \mathcal{L}(\chi), \quad (2.2.3)$$

where $\mathcal{L}(\chi)$ denotes the G -linearized invertible sheaf on G/H associated with the character χ . Moreover, the G -module $\Gamma(G/H, \mathcal{L}(\chi))$ is isomorphic to $k[G]_\chi^{(H)}$ (the space of right H -eigenvectors of weight χ); see, e.g., [25, Section 2.1] for these facts.

Proposition 2.2.1.

(1) *The sequence*

$$0 \rightarrow \mathbb{Z}^r \xrightarrow{u} \mathrm{Pic}(X) \xrightarrow{i^*} \mathrm{Pic}(G/H) \rightarrow 0 \quad (2.2.4)$$

is exact, where $u(x_1, \dots, x_r) := \sum_{i=1}^r x_i [X_i]$.

- (2) *The canonical section $s_D \in \Gamma(X, \mathcal{O}_X(D))$ is a B -eigenvector of weight ω_D ; the canonical section $s_i \in \Gamma(X, \mathcal{O}_X(X_i))$ is G -invariant.*
 (3) *The natural map $\mathcal{X}(B) \rightarrow \mathrm{Pic}^B(X_B)$ is an isomorphism. Via the inverse isomorphism, the restriction of $\mathcal{O}_X(D)$ (respectively $\mathcal{O}_X(X_i)$) is mapped to ω_D (respectively γ_i).*
 (4) *The product of restriction maps*

$$\pi : \mathrm{Pic}(X) = \mathrm{Pic}^G(X) \rightarrow \mathrm{Pic}^B(X_B) \times \mathrm{Pic}^G(G/H) \cong \mathcal{X}(B) \times \mathcal{X}(H)$$

is injective, and its image is $\mathcal{X}(B) \times_{\mathcal{X}(B \cap H)} \mathcal{X}(H)$. Together with the isomorphism (2.2.3), this identifies (2.2.4) with the exact sequence

$$0 \rightarrow \Lambda(X) = \mathcal{X}(B)^{B \cap H} = \bigoplus_{i=1}^r \mathbb{Z} \gamma_i \rightarrow \mathbb{Z}^D = \mathcal{X}(B) \times_{\mathcal{X}(B \cap H)} \mathcal{X}(H) \rightarrow \mathcal{X}(H) \rightarrow 0, \quad (2.2.5)$$

where the map on the right is the second projection. Moreover, each $[X_i]$ is identified with the spherical root γ_i .

Proof. (1) Since $G/H = X \setminus \bigcup_{i=1}^r X_i$, the map i^* is surjective and its kernel is generated by the classes $[X_1], \dots, [X_r]$. They are linearly independent: if $f \in k(X)$ has zeroes and poles along X_1, \dots, X_r only, then f must be G -semi-invariant, and hence constant.

(2) The restriction of s_D to G/H equals

$$f_D \in k[G]_{\chi_D}^{(H)} \cong \Gamma(G/H, \mathcal{L}(\chi_D)) = \Gamma(G/H, \mathcal{O}_X(D)),$$

a B -eigenvector of weight ω_D . The G -invariance of s_i is readily checked.

(3) The first assertion follows from the fact that $k[X_B]$ is a UFD and $k[X_B]^* = k^*$. For the second assertion, observe that s_D yields a trivialization of $\mathcal{O}_X(D)$ over X_B . Likewise, f_i yields a trivialization of $\mathcal{O}_X(X_i)$ over X_B .

(4) π maps any class $[D]$ to the pair (ω_D, χ_D) , and any class $[X_i]$, to $(\gamma_i, 0)$. Together with Lemma 2.1.1, this implies the assertions. \square

Next we determine the sections of the positive multiples of boundary divisors.

Definition 2.2.2. We say that a prime divisor D in X is *fixed* if the global sections of any positive integral multiple nD are just the scalar multiples of the canonical section $s_{nD} = s_D^n$. Equivalently, any regular function on $X \setminus D$ is constant.

Theorem 2.2.3. *Each boundary divisor X_i is either fixed or base-point-free. In the latter case, the spherical root γ_i is a dominant weight, and there is an isomorphism of G -modules*

$$\Gamma(X, \mathcal{O}_X(nX_i)) \cong \bigoplus_{m=0}^n V(m\gamma_i). \quad (2.2.6)$$

Moreover, the G -variety

$$X' := \text{Proj} \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nX_i))$$

is a wonderful variety of rank 1; its boundary divisor X'_i is ample, and $X_i = \varphi^{-1}(X'_i)$ where $\varphi: X \rightarrow X'$ is the natural morphism.

Proof. Let $f \in k(X)^{(B)}$ such that the divisor $\text{div}(f) + nX_i$ is effective. Equivalently, $v_D(f) \geq 0$ for all $D \in \mathcal{D}$, $v_j(f) \geq 0$ for all $j \neq i$, and $v_i(f) + n \geq 0$. By Lemma 2.1.2, it follows that $v_j(f) = 0$ for all $j \neq i$, that is, f is a scalar multiple of some power f_i^{-m} , where $m \in \mathbb{Z}$. Then, by (2.2.1),

$$\text{div}(f) + nX_i = (n - m)X_i + m \sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle D.$$

Moreover, there exists $D \in \mathcal{D}$ such that $\langle D, \gamma_i \rangle > 0$, since X_i cannot be linearly equivalent to the opposite of an effective divisor. Thus, $0 \leq m \leq n$.

If there exists $D \in \mathcal{D}$ such that $\langle D, \gamma_i \rangle < 0$, then $m = 0$, i.e., f is constant: X_i is fixed.

Otherwise, X_i is base-point-free, $\gamma_i = \sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle \omega_D$ is dominant, and the B -eigenvectors in $\Gamma(X, \mathcal{O}_X(nX_i))$ are exactly the scalar multiples of the $s_i^n f_i^{-m}$, where $0 \leq m \leq n$. This implies (2.2.6). As a consequence, the B -eigenvectors in the G -algebra $\bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nX_i))$ are exactly the monomials in s_i and $s_i f_i^{-1}$.

By construction, X' is a normal projective G -variety equipped with a G -equivariant morphism $\varphi: X \rightarrow X'$ such that the induced map $\mathcal{O}_{X'} \rightarrow \varphi_* \mathcal{O}_X$ is an isomorphism; in particular, φ is surjective with connected fibers. Moreover, $X_i = \varphi^{-1}(X'_i)$ where X'_i is an ample G -stable divisor on X' . Together with the preceding argument, it follows that X' is spherical of rank 1 with boundary X'_i ; hence X' is wonderful. \square

The G -varieties X' that appear in Theorem 2.2.3 are precisely the wonderful G -varieties of rank 1 having an ample boundary divisor. Equivalently, their open orbit G/H' is affine (indeed, the complement of an ample divisor is affine; conversely, if G/H' is affine, then its complement is ample as follows, e.g., from Theorem 2.2.3); in other words, the isotropy group H' is reductive. These varieties have been classified by Akhiezer [2, Theorem 4] over the complex numbers; he showed that they correspond bijectively to compact isotropic Riemannian manifolds (see also [9] for a more algebraic approach). It turns out that the varieties X' are all homogeneous under

their full automorphism group; this is deduced in [2] from the classification, while an a priori proof is obtained in [6, Corollary 4.1.3]. We shall show in Theorem 2.4.1 that $\varphi: X \rightarrow X'$ is a homogeneous fibration with respect to an overgroup of G .

Example 2.2.4. Let $X = \bar{G}_{\text{ad}}$ as in Example 2.1.3. Then the Picard group may be identified with the weight lattice of G . The classes of colors (respectively boundary divisors) are then identified with the fundamental weights (respectively simple roots). The nef cone is the positive Weyl chamber. The fixed boundary divisors correspond to those simple roots α_i that are dominant. Equivalently, α_i is orthogonal to all other simple roots, that is, the image of the associated morphism $\text{SL}_2 \rightarrow G$ yields a direct factor of G_{ad} isomorphic to PSL_2 . Then the variety X' of Theorem 2.2.3 is the wonderful compactification of PSL_2 , i.e., the projectivization $\mathbb{P}(M_2)$ of the space of 2×2 matrices; it is a direct factor of X , and the map φ is just the corresponding projection.

2.3. The cone of effective divisors

Denote by $\text{Eff}(X)_{\mathbb{R}}$ the cone in $\text{Pic}(X)_{\mathbb{R}} \cong \mathbb{R}^{\mathcal{D}}$ generated by the classes of effective divisors. This *cone of effective divisors* is generated by the classes of the colors and of the boundary divisors; in particular, it is a rational polyhedral convex cone, which contains no line as X is complete. The inclusion $\text{Eff}(X) \subseteq \text{Eff}_{\mathbb{R}}(X) \cap \text{Pic}(X)$ may be strict; in other words, a non-effective divisor may have an effective non-zero multiple, see Example 2.3.5.

We first describe the intersection of the cone of effective divisors with the subspace spanned by the classes of boundary divisors (these are linearly independent by Proposition 2.2.1).

Lemma 2.3.1. $\text{Eff}(X)_{\mathbb{R}} \cap \text{Span}([X_1], \dots, [X_r]) = \text{Cone}([X_1], \dots, [X_r])$.

Proof. It suffices to show that if x_1, \dots, x_r are integers such that $\sum_{i=1}^r x_i [X_i]$ is effective, then they are all non-negative. There exists $f \in k(X)^{(B)}$ such that

$$\sum_{i=1}^r x_i X_i = \text{div}(f) + \sum_{i=1}^r y_i X_i + \sum_{D \in \mathcal{D}} z_D D,$$

where $y_i, z_D \geq 0$. So $v_D(f) \leq 0$ for all D , and hence $v_i(f) \geq 0$ for all i by Lemma 2.1.2. Hence $x_i = v_i(f) + y_i \geq 0$. \square

We now turn to the description of the extremal rays of $\text{Eff}(X)_{\mathbb{R}}$. Clearly, any such ray is generated by the class of a boundary divisor or a color; moreover, the classes of colors are pairwise non-proportional, and the same holds for the classes of boundary divisors. But a boundary divisor may well be proportional to a color, e.g., when $\text{Pic}(X)$ has rank 1. So we may obtain three types of extremal rays, considered separately in the following lemmas.

Lemma 2.3.2. *For a boundary divisor X_i , the following conditions are equivalent:*

- (1) $[X_i]$ generates an extremal ray of $\text{Eff}(X)_{\mathbb{R}}$ which does not contain the class of a color.
- (2) X_i is fixed.
- (3) $\langle D, \gamma_i \rangle < 0$ for some $D \in \mathcal{D}$.

Proof. (1) is equivalent to the non-existence of $f \in k(X)^{(B)} \setminus k^*$ such that $\operatorname{div}(f) + nX_i$ is effective for some positive integer n . In turn, this is equivalent to $k[X \setminus X_i]^{(B)} = k$. But since $k[X \setminus X_i]$ is a rational G -module, the latter condition amounts to $k[X \setminus X_i] = k$. Thus, (1) \Leftrightarrow (2).

(2) \Leftrightarrow (3) follows from (2.2.2) and Theorem 2.2.3. \square

Lemma 2.3.3. *For a color D , the following conditions are equivalent:*

- (1) $[D]$ generates an extremal ray of $\operatorname{Eff}(X)_{\mathbb{R}}$ which does not contain the class of a boundary divisor.
- (2) There exists a G -equivariant morphism

$$\varphi: X \rightarrow G/P,$$

where $P \supseteq H$ is a maximal parabolic subgroup of G , such that D is the preimage of the Schubert divisor (the unique B -stable prime divisor) in G/P .

Proof. As in the proof of Lemma 2.3.2, (1) is equivalent to the condition that any B -eigenvector in $\Gamma(X, \mathcal{O}_X(nD))$ is a scalar multiple of s_D^n , for all $n \geq 1$. In turn, this is equivalent to the simplicity of each G -module $\Gamma(X, \mathcal{O}_X(nD))$, and hence to the condition that the G -variety

$$X' := \operatorname{Proj} \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nD))$$

is isomorphic to some G/P . Since D is base-point-free, X comes with a G -morphism $\varphi: X \rightarrow X'$, and hence we may assume that $P \supseteq H$. Moreover, $\varphi(D)$ is a prime B -stable ample divisor in G/P , so that the parabolic subgroup P is maximal. \square

Lemma 2.3.4. *For a boundary divisor X_i and a color D , the following conditions are equivalent:*

- (1) $[X_i]$ is a multiple of $[D]$.
- (2) There exists a surjective G -equivariant morphism

$$\varphi: X \rightarrow X'$$

with connected fibers, where X' is a wonderful G -variety of rank 1 having a unique color D' . Moreover, $D = \varphi^{-1}(D')$ and $X_i = \varphi^{-1}(X'_i)$, where X'_i denotes the unique boundary divisor of X' .

If (1) or (2) holds, then $[X_i]$ generates an extremal ray of $\operatorname{Eff}(X)_{\mathbb{R}}$.

Proof. (1) \Leftrightarrow (2) follows from Theorem 2.2.3. It also follows that $\operatorname{div}(f_i) = X_i - aD$ for some positive integer a , and that the effective B -stable divisors linearly equivalent to a given positive multiple nX_i are exactly the $(n - m)X_i + maD$, where $m = 0, \dots, n$. This implies the final assertion. \square

The varieties X' that appear in Lemma 2.3.4 are exactly the wonderful varieties of rank 1 and Picard group \mathbb{Z} (in particular, their boundary divisor is ample). Their classification follows from that in [2, Theorem 4].

Example 2.3.5. If $X = \bar{G}_{\text{ad}}$, then the extremal rays of $\text{Eff}(X)_{\mathbb{R}}$ are generated by the simple roots $\alpha_1, \dots, \alpha_r$, while those of $\text{Nef}(X)_{\mathbb{R}}$ are generated by the fundamental weights $\omega_1, \dots, \omega_r$.

If G is simple of type B_2 , then $\omega_1 - \omega_2 = \frac{1}{2}\alpha_1$, so that the colors D_1, D_2 satisfy $[D_1] - [D_2] = \frac{1}{2}[X_1]$. Thus, $E := D_1 - D_2$ is a non-effective (integral) divisor such that $2E$ is effective.

Returning to an arbitrary wonderful G -variety X , if the isotropy group H is not contained in any proper parabolic subgroup of G , then the cone $\text{Eff}(X)_{\mathbb{R}}$ is generated by the classes of boundary divisors by Lemma 2.3.3. Together with Proposition 2.2.1, it follows that $\text{Eff}(X)_{\mathbb{R}}$ is simplicial. But this need not hold in general, as shown by the following example.

Example 2.3.6. In the product

$$\mathbb{P}^2 \times \mathbb{P}^2 \times \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2,$$

where $\check{\mathbb{P}}^2$ denotes the projective plane of lines in \mathbb{P}^2 , let X be the closed subvariety defined by the incidence conditions

$$p_1 \in d_1, \quad p_2 \in d_1, \quad p_2 \in d_2,$$

where $p_1, p_2 \in \mathbb{P}^2$ and $d_1, d_2 \in \check{\mathbb{P}}^2$. Then X is stable under the diagonal action of $G := \text{SL}_3$ and one easily checks that X is a wonderful G -variety with boundary divisors X_1 (the locus where $p_1 = p_2$) and X_2 (where $d_1 = d_2$). Moreover, X has 4 colors, the pull-backs of the B -stable lines in $\mathbb{P}^2, \check{\mathbb{P}}^2$ via the 4 projections. One also checks (either directly or by using the above results) that the cone $\text{Eff}(X)_{\mathbb{R}}$ in $\text{Pic}(X)_{\mathbb{R}} \cong \mathbb{R}^4$ has 6 extremal rays, corresponding bijectively to all the B -stable prime divisors in X .

2.4. The connected automorphism group

Denote by $\text{Aut}(X)$ the automorphism group of X , and by $\text{Aut}^0(X)$ its connected component containing the identity. Since X is complete and $\text{Pic}(X)$ is discrete, the group $\text{Aut}^0(X)$ is linear algebraic and its Lie algebra is the space of global vector fields, $\Gamma(X, \mathcal{T}_X)$ (see, e.g., [31]).

The subsheaf $\mathcal{S}_X \subseteq \mathcal{T}_X$ consisting of those vector fields that preserve the boundary is a locally free sheaf, called the logarithmic tangent sheaf; it contains the image of the natural map $\mathcal{O}_X \otimes \mathfrak{g} \rightarrow \mathcal{T}_X$, where \mathfrak{g} denotes the Lie algebra of G . The subspace $\Gamma(X, \mathcal{S}_X) \subseteq \Gamma(X, \mathcal{T}_X)$ is the Lie algebra of the subgroup $\text{Aut}^0(X, \partial X) \subseteq \text{Aut}^0(X)$, and this subgroup preserves all the G -orbits in X .

In fact, the induced map $\mathcal{O}_X \otimes \mathfrak{g} \rightarrow \mathcal{S}_X$ is surjective, and we have an exact sequence

$$0 \rightarrow \Gamma(X, \mathcal{S}_X) \rightarrow \Gamma(X, \mathcal{T}_X) \rightarrow \bigoplus_{i=1}^r \Gamma(X_i, \mathcal{N}_{X_i/X}) \rightarrow 0, \quad (2.4.1)$$

where $\mathcal{N}_{X_i/X} = \mathcal{O}_{X_i}(X_i)$ denotes the normal sheaf. Moreover, the sequence

$$0 \rightarrow k \xrightarrow{s_i} \Gamma(X, \mathcal{O}_X(X_i)) \rightarrow \Gamma(X_i, \mathcal{N}_{X_i/X}) \rightarrow 0 \quad (2.4.2)$$

is exact for $i = 1, \dots, r$ (for these results, see [6]).

Together with Theorem 2.2.3, it follows that the G -module $\Gamma(X_i, \mathcal{N}_{X_i/X})$ is either 0 (if X_i is fixed), or simple with highest weight γ_i (if X_i is base-point-free).

Theorem 2.4.1.

- (1) *The fixed divisors in X are exactly the boundary divisors X_i such that $\Gamma(X, \mathcal{O}_X(X_i)) = ks_i$. They are all stable under $\text{Aut}^0(X)$.*
- (2) *The base-point-free boundary divisors are all moved by $\text{Aut}^0(X)$. For any such divisor X_i and any closed subgroup $G' \subseteq \text{Aut}^0(X)$ containing the image of G , either X_i is G' -stable or it contains no G' -orbit; in particular, X_i contains no orbit of $\text{Aut}^0(X)$. The morphism $\varphi: X \rightarrow X'$ associated with X_i as in Theorem 2.2.3 is equivariant under $\text{Aut}^0(X)$, which acts transitively on X' .*
- (3) *The extremal rays of $\text{Eff}(X)_{\mathbb{R}}$ consist of the rays generated by the fixed divisors, and of those generated by the pull-backs of Schubert divisors under $\text{Aut}^0(X)$ -equivariant morphisms $\varphi: X \rightarrow X'$, where X' is the quotient of $\text{Aut}^0(X)$ by a maximal parabolic subgroup. Both types of rays are disjoint.*

Proof. (1) Let D be a fixed divisor; then D is stable under $\text{Aut}^0(X)$, since $\xi \cdot s_D$ is a scalar multiple of s_D for any $\xi \in \Gamma(X, \mathcal{T}_X)$. In particular, D is a boundary divisor; together with Theorem 2.2.3, this implies the assertions.

(2) Let X_i be a base-point-free boundary divisor. Then X_i is not stable under $\text{Aut}^0(X)$, by (2.4.1) and (2.4.2). Moreover, some positive power of $\mathcal{O}_X(X_i)$ admits a linearization for the group $\text{Aut}^0(X)$, see [25, Section 2.4]. Therefore, this group acts on X' so that φ is equivariant. Let $G' \subseteq \text{Aut}^0(X)$ be a closed subgroup containing the image of G and moving X_i . Then G' acts on X' and moves X'_i , so that X' is a unique G' -orbit. Thus, X_i contains no G' -orbit.

(3) follows from (2) together with Lemmas 2.3.2, 2.3.3 and 2.3.4. \square

Theorem 2.4.2.

- (1) *Let I be any subset of $\{1, \dots, r\}$ containing the indices of the fixed divisors and let $\text{Aut}^0(X, \partial_I X)$ be the stabilizer in $\text{Aut}^0(X)$ of the X_i , $i \in I$. Then $\text{Aut}^0(X, \partial_I X)$ is semi-simple and X is a wonderful variety under that group, with boundary divisors being the X_i , $i \in I$.*
- (2) *Let G' be a closed connected subgroup of $\text{Aut}^0(X)$ containing the image of G . Then G' is semi-simple and the G' -variety X is wonderful; its set of colors (relative to some Borel subgroup containing the image of B) is exactly \mathcal{D} . Moreover, there are only finitely many such intermediate subgroups G' .*

In particular, $\text{Aut}^0(X)$ is semi-simple and X is a wonderful variety under that group, with boundary divisors being the fixed divisors, and with set of colors being \mathcal{D} .

Proof. The Lie algebra of $\text{Aut}^0(X, \partial_I X)$ is the kernel of the natural map

$$\Gamma(X, \mathcal{T}_X) \rightarrow \bigoplus_{i \in I} \Gamma(X_i, \mathcal{N}_{X_i/X}).$$

By (2.4.1) and Theorem 2.4.1, it follows that $\text{Aut}^0(X, \partial_I X)$ moves each X_j , $j \notin I$.

The G -variety

$$X_I := \bigcap_{i \in I} X_i$$

is wonderful and stable under $\text{Aut}^0(X, \partial_I X)$. Moreover, any proper G -stable closed subvariety of X_I is contained in X_j for some $j \notin I$, and hence contains no orbit of $\text{Aut}^0(X, \partial_I X)$, by Theorem 2.4.1 again. It follows that X_I is a unique orbit of $\text{Aut}^0(X, \partial_I X)$.

Let G_I be a Levi subgroup of $\text{Aut}^0(X, \partial_I X)$ containing the image of G and let B_I be a Borel subgroup of G_I containing the image of B . Then G_I is semi-simple (since the G -automorphism group of X is finite) and acts transitively on the projective $\text{Aut}^0(X, \partial_I X)$ -orbit X_I . Clearly, the G_I -variety X is spherical, X_I is its unique closed orbit, and its B_I -stable prime divisors are among the $D \in \mathcal{D}$ and X_1, \dots, X_r . By Theorem 2.4.1, the boundary divisors are either G_I -stable or contain no G_I -orbit. Thus, the G_I -variety X is toroidal, and hence wonderful by [21, Section 7]. Its boundary divisors are those X_i that contain X_I , i.e., such that $i \in I$. They are all stable under $\text{Aut}^0(X, \partial_I X)$, so that this group is semi-simple by Lemma 2.4.3. In other words, $G_I = \text{Aut}^0(X, \partial_I X)$. This completes the proof of (1).

For (2), let I be the set of G' -stable boundary divisors. Arguing as above, we see that X_I is a unique orbit of G' , and hence of a Levi subgroup $G'' \subseteq G'$ containing the image of G . Moreover, G'' is semi-simple and X is wonderful for that group, with boundary divisors indexed by I . Applying Lemma 2.4.3 again, we obtain that $G'' = G'$. The finiteness of the set of subgroups G' follows from Lemma 2.4.4.

To complete the proof, it suffices to show that $X_B = X_{B'}$ for an appropriate Borel subgroup B' of G' containing the image of B . By [27, Section 1], X_B is the Bialynicki–Birula cell $X(\lambda, z)$ where $z \in X_B$ is the unique T -fixed point, and λ is a general one-parameter subgroup of T such that the associated parabolic subgroup $G(\lambda)$ equals B . This defines a parabolic subgroup $G'(\lambda)$ of G' , with Levi subgroup being the centralizer $C_{G'}(\lambda) = C_{G'}(T)$.

We claim that X_B is stable under $G'(\lambda)$. Given $x \in X_B$ and $g' \in G'(\lambda)$,

$$\lambda(t)g' \cdot x \rightarrow g'_0 \cdot z \quad \text{as } t \rightarrow 0,$$

where $g'_0 := \lim_{t \rightarrow 0} \lambda(t)g'\lambda(t^{-1})$ is in $C_{G'}(\lambda)$. Moreover, $C_{G'}(\lambda)$ is connected, and $C_{G'}(\lambda) \cdot z$ consists of T -fixed points. The latter being isolated, it follows that $g'_0 \cdot z = z$ which proves the claim.

Let B' be a Borel subgroup of $G'(\lambda)$ containing the image of B . Then X_B is stable under B' , which implies the desired equality. \square

Lemma 2.4.3. *Any closed connected subgroup of $\text{Aut}^0(X, \partial X)$ which contains G is semi-simple.*

Proof. Otherwise, there exists a non-trivial closed connected unipotent subgroup A of $\text{Aut}^0(X, \partial X)$ which is normalized by G ; we may further assume that A is abelian. The semi-direct product $\tilde{G} := G \cdot A$ acts on X with the same orbits as those of G . Moreover, the invariant subgroup A^G is trivial, since $\text{Aut}_G(X)$ is finite.

Let $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$ be the Lie algebra of \tilde{G} and let S_X be the \tilde{G} -linearized vector bundle on X with sheaf of local sections S_X . Then the map $\mathfrak{g} \rightarrow \Gamma(X, S_X)$ yields a G -equivariant morphism

$$\mu: S_X^* \rightarrow \mathfrak{g}^*,$$

the localized moment map of [23]. This morphism is proper, and so is the analogous \tilde{G} -equivariant morphism

$$\tilde{\mu} : S_X^* \rightarrow \tilde{\mathfrak{g}}^*.$$

Thus, both images $\text{Im}(\mu)$, $\text{Im}(\tilde{\mu})$ are closed, and the projection

$$p : \tilde{\mathfrak{g}}^* = \mathfrak{g}^* \oplus \mathfrak{a}^* \rightarrow \mathfrak{g}^*$$

restricts to a finite G -morphism, also denoted by $p : \text{Im}(\tilde{\mu}) \rightarrow \text{Im}(\mu)$.

Consider the quotient map

$$q : \mathfrak{g}^* \rightarrow \mathfrak{g}^* // G.$$

We claim that the composite morphism

$$q \circ \mu : S_X^* \rightarrow \mathfrak{g}^* \rightarrow \mathfrak{g}^* // G$$

is \tilde{G} -invariant. Indeed, by [23, Theorem 6.1], any regular function on S_X^* arising from $q \circ \mu$ is the symbol of a G -invariant differential operator on X which is completely regular in the sense of [23, p. 262]. Thus, this operator commutes with the action of $\tilde{\mathfrak{g}} \subseteq \Gamma(X, \mathcal{S}_X)$, by [23, Corollary 7.6]. This implies our claim.

By that claim, the composite morphism

$$\pi := q \circ p : \text{Im}(\tilde{\mu}) \rightarrow \text{Im}(\mu) \rightarrow \mathfrak{g}^* // G$$

is \tilde{G} -invariant. Moreover, since p is finite and any fiber of q contains only finitely many G -orbits, any fiber of π contains only finitely many G -orbits as well. It follows that each general \tilde{G} -orbit in $\text{Im}(\tilde{\mu})$ contains an open G -orbit. In other words, the equality $\tilde{\mathfrak{g}} \cdot f = \mathfrak{g} \cdot f$ holds for any f in a dense subset of $\text{Im}(\tilde{\mu})$. By [11, Theorem 2.2], this implies that $f \in \mathfrak{g}^*$, i.e., $\text{Im}(\tilde{\mu}) \subseteq \mathfrak{g}^*$. In other words, the map $\mathfrak{a} \rightarrow \Gamma(X, \mathcal{S}_X)$ is zero, a contradiction. \square

Lemma 2.4.4. *Let $G_1 \subseteq G_2$ be semi-simple groups such that any closed connected subgroup $G' \subseteq G_2$ containing G_1 is semi-simple. Then the number of such intermediate subgroups is finite.*

Proof. This result is certainly known; for lack of a reference, we provide an argument. Consider the semi-simple Lie algebras $\mathfrak{g}_1 \subseteq \mathfrak{g}' \subseteq \mathfrak{g}_2$. By [34], the orbit $G_2 \cdot \mathfrak{g}'$ is open in the variety of Lie subalgebras of \mathfrak{g}_2 , and hence the number of G_2 -conjugacy classes of intermediate subgroups G' is finite. Moreover, the centralizer of G_1 in G_2 is finite. By [35], it follows that any G_2 -orbit contains only finitely many G_1 -fixed points. Hence any conjugacy class contains only finitely many overgroups of G_1 . \square

Example 2.4.5. We determine the automorphism group of $X = \tilde{G}_{\text{ad}}$. Write

$$G = (\text{SL}_2)^n \times G',$$

where G' contains no direct factor isomorphic to SL_2 . Then

$$X \cong \mathbb{P}(M_2)^n \times X' \cong (\mathbb{P}^3)^n \times X',$$

where $X' := \bar{G}'_{\text{ad}}$. Thus,

$$\text{Aut}^0(X) \cong (\text{PSL}_4)^n \times \text{Aut}^0(X').$$

Since all boundary divisors of X' are fixed, the semi-simple group $\text{Aut}^0(X')$ stabilizes all the $G' \times G'$ -orbits; as a consequence, it acts faithfully on the closed orbit. The latter is isomorphic to $G'/B' \times G'/B'$, and $\text{Aut}^0(G'/B' \times G'/B') = G'_{\text{ad}} \times G'_{\text{ad}}$ by [16], so that

$$\text{Aut}^0(X') = G'_{\text{ad}} \times G'_{\text{ad}}$$

as well.

The full automorphism group $\text{Aut}(X)$ acts on the nef monoid and permutes its generators $[D]$, $D \in \mathcal{D}$; it stabilizes the subset of those $[D]$ that generate an extremal ray of $\text{Eff}(X)_{\mathbb{R}}$, i.e., the pull-backs of hyperplanes under the n projections $X \rightarrow \mathbb{P}^3$. It follows that

$$\text{Aut}(X) \cong \text{Aut}((\mathbb{P}^3)^n) \times \text{Aut}(X').$$

Moreover, $\text{Aut}((\mathbb{P}^3)^n)$ is the semi-direct product of $(\text{PSL}_4)^n$ by the symmetric group S_n permuting the copies of \mathbb{P}^3 .

To determine $\text{Aut}(X')$, we consider its action by conjugation on its connected component $G'_{\text{ad}} \times G'_{\text{ad}}$. This action is faithful, since $\text{Aut}_{G' \times G'}(X')$ is trivial. Moreover, any $\varphi \in \text{Aut}(X')$ sends $\text{diag}(G'_{\text{ad}})$ to a conjugate subgroup in $G'_{\text{ad}} \times G'_{\text{ad}}$, since φ preserves the open orbit $X'^0 \cong (G'_{\text{ad}} \times G'_{\text{ad}})/\text{diag}(G'_{\text{ad}})$. It follows that the group $\text{Aut}(X')$ is generated by $G'_{\text{ad}} \times G'_{\text{ad}}$ together with the flip $(g, h) \mapsto (h, g)$ and with the automorphisms (φ, φ) , where φ arises from an automorphism of the Dynkin diagram of G' .

3. The total coordinate ring of a wonderful variety

3.1. Definitions and basic properties

The notation and assumptions are as in Section 2; in particular, X denotes a wonderful variety under the simply-connected semi-simple group G , and \mathcal{D} denotes its set of colors.

Consider the sum

$$\bigoplus_{(n_D) \in \mathbb{Z}^{\mathcal{D}}} \mathcal{O}_X \left(\sum_{D \in \mathcal{D}} n_D D \right).$$

This is a sheaf of \mathcal{O}_X -algebras, graded by $\mathbb{Z}^{\mathcal{D}}$, and compatibly G -linearized. It defines a scheme \widehat{X} equipped with a morphism

$$p: \widehat{X} \rightarrow X$$

and with an action of the group

$$\widetilde{G} := G \times \mathbb{G}_m^{\mathcal{D}},$$

where $\mathbb{G}_m^{\mathcal{D}} \cong \text{Hom}(\text{Pic}(X), \mathbb{G}_m)$ denotes the torus with character group $\mathbb{Z}^{\mathcal{D}} \cong \text{Pic}(X)$. The morphism p is G -equivariant and is a principal $\mathbb{G}_m^{\mathcal{D}}$ -bundle: the *universal torsor* over X . In particular, \widehat{X} is a non-singular spherical variety under the connected reductive group \widetilde{G} .

By [5, Proposition 3.10], the variety \widehat{X} is quasi-affine. This may also be seen directly: let \mathcal{L} be a very ample invertible sheaf on X . Then the sheaves $\mathcal{L} \otimes \mathcal{O}_X(D)$, $D \in \mathcal{D}$ are all very ample, and form a basis of $\text{Pic}(X)$. Consider the corresponding projective embeddings $X \hookrightarrow \mathbb{P}_D$, and the tautological principal \mathbb{G}_m -bundles $\widehat{\mathbb{P}}_D \rightarrow \mathbb{P}_D$. Then each $\widehat{\mathbb{P}}_D$ is the complement of a point in an affine space, and \widehat{X} is the pull-back of the product bundle under the diagonal embedding $X \rightarrow \prod_{D \in \mathcal{D}} \mathbb{P}_D$.

Next let

$$R(X) := \Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}}) = \bigoplus_{(n_D) \in \mathbb{Z}^{\mathcal{D}}} \Gamma\left(X, \mathcal{O}_X\left(\sum_{D \in \mathcal{D}} n_D D\right)\right).$$

This is a $\mathbb{Z}^{\mathcal{D}}$ -graded k -algebra called the *total coordinate ring* of X . The set of degrees of its homogeneous elements is the monoid $\text{Eff}(X)$, which generates the rational polyhedral cone $\text{Eff}(X)_{\mathbb{R}}$ containing no line. As a consequence, $R(X)$ admits a coarser grading by the non-negative integers, such that $R(X)_0 = \Gamma(X, \mathcal{O}_X) = k$. It follows that $R(X)^* = k^*$.

The action of \widetilde{G} on \widehat{X} yields an action on $R(X)$. Since \widehat{X} is spherical, the k -algebra $R(X)$ is finitely generated (see [22]) and normal. Thus,

$$\widetilde{X} := \text{Spec } R(X)$$

is a normal affine variety equipped with a \widetilde{G} -equivariant morphism

$$\iota: \widehat{X} \rightarrow \widetilde{X}.$$

By [5], ι is an open immersion. Again, this may be seen directly: let \mathcal{L} be as above, then \widehat{X} is covered by the affine open subsets $\widehat{X}_s := p^{-1}(X_s)$, where s is a non-zero global section of \mathcal{L} . Moreover, $\widehat{X}_s = \iota^{-1}(\widetilde{X}_s)$ where s is regarded as a homogeneous element of $R(X)$, and the induced homomorphism $R(X)[s^{-1}] \rightarrow k[\widehat{X}_s] = k[p^{-1}(X_s)]$ is an isomorphism, since each space $\Gamma(X_s, \mathcal{O}_X(\sum n_D D))$ is the union of its subspaces $\Gamma(X, \mathcal{L}^N(\sum n_D D))s^{-N}$, where $N \geq 0$.

Since \widehat{X} and \widetilde{X} have the same algebra of regular functions, the closed subset $\widetilde{X} \setminus \iota(\widehat{X})$ has codimension at least 2 in \widetilde{X} . Thus, \widetilde{X} is a spherical \widetilde{G} -variety with open \widetilde{G} -orbit

$$\widetilde{X}^0 := p^{-1}(X^0).$$

Moreover,

$$\widetilde{X}_B^0 := p^{-1}(X_B^0)$$

is the open orbit of

$$\widetilde{B} := B \times \mathbb{G}_m^{\mathcal{D}}$$

(a Borel subgroup of \tilde{G}), and

$$\tilde{X} \setminus \tilde{X}_{\tilde{B}}^0 = \tilde{X}_1 \cup \dots \cup \tilde{X}_r \cup \bigcup_{D \in \mathcal{D}} \tilde{D}, \quad (3.1.1)$$

where \tilde{X}_i denotes the closure in \tilde{X} of the prime divisor $p^{-1}(X_i)$, and likewise for \tilde{D} . Note also that \tilde{X} contains a fixed point of \tilde{G} , since $R(X)$ admits a positive \tilde{G} -invariant grading.

By [5, Proposition 8.4] or [18, Corollary 1.2], $R(X)$ is a UFD. This may also be seen directly, by showing that the divisor class group of \tilde{X} is trivial. Indeed, the open subset $\tilde{X}_{\tilde{B}}^0$ has a trivial divisor class group, and the ideal of each irreducible component of its complement is generated by the corresponding canonical section.

We now describe the invariant subalgebras $R(X)^U$ and $R(X)^G$.

Proposition 3.1.1.

- (1) *The algebra $R(X)^U$ is freely generated by the canonical sections s_1, \dots, s_r and the s_D , $D \in \mathcal{D}$. Each s_D is an eigenvector of $\mathbb{G}_m^{\mathcal{D}}$ of weight the D -component*

$$\varepsilon_D : \mathbb{G}_m^{\mathcal{D}} \rightarrow \mathbb{G}_m.$$

Moreover, each s_i is a $\mathbb{G}_m^{\mathcal{D}}$ -eigenvector of weight $\sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle \varepsilon_D$.

- (2) *The subalgebra $R(X)^G$ is freely generated by s_1, \dots, s_r .*
 (3) *$R(X)$ is a free module over $R(X)^G$.*
 (4) *The quotient morphism*

$$q : \tilde{X} = \operatorname{Spec} R(X) \rightarrow \operatorname{Spec} R(X)^G =: \tilde{X} // G \cong \mathbb{A}^r$$

is flat and its (scheme-theoretic) fibers are normal varieties.

Proof. (1) Any element of $R(X)^U$ can be uniquely decomposed into a sum of \tilde{B} -eigenvectors. We claim that any such eigenvector σ is a monomial in s_1, \dots, s_r and the s_D , $D \in \mathcal{D}$, with uniquely determined exponents. Indeed, σ is a B -eigenvector in some space

$$\Gamma\left(X, \mathcal{O}_X\left(\sum_{D \in \mathcal{D}} n_D D\right)\right).$$

Its divisor of zeroes may be written uniquely as

$$\operatorname{div}(\sigma) = \sum_{D \in \mathcal{D}} \operatorname{ord}_D(\sigma) D + \sum_{i=1}^r \operatorname{ord}_{X_i}(\sigma) X_i.$$

Thus, σ is a scalar multiple of the monomial $\prod_{D \in \mathcal{D}} s_D^{\operatorname{ord}_D(\sigma)} \cdot \prod_{i=1}^r s_i^{\operatorname{ord}_{X_i}(\sigma)}$ and of no other monomial. This proves the claim and, in turn, the assertion on $R(X)^U$.

Clearly, each s_D has weight ε_D ; by (2.2.2), this implies the assertion on the weight of s_i .

(2) Likewise, each \tilde{G} -eigenvector in $R(X)$ is a monomial in s_1, \dots, s_r with uniquely determined exponents.

(3) Let M be the G -submodule of $R(X)$ generated by the monomials in the s_D , $D \in \mathcal{D}$. Then M^U is just the polynomial ring $k[s_D, D \in \mathcal{D}]$ (but M is generally not a subring of $R(X)$). Consider the multiplication map

$$m: R(X)^G \otimes M \rightarrow R(X).$$

This is a morphism of G -modules, which restricts to an isomorphism

$$(R(X)^G \otimes M)^U = R(X)^G \otimes M^U \cong R(X)^U \quad (3.1.2)$$

by (1). Thus, m is an isomorphism.

(4) By (3), q is flat. Let F be a fiber at a closed point of \tilde{X}/G , then F is an affine G -scheme and the restriction map $M^U \rightarrow k[F]^U$ is an isomorphism by (3.1.2). Thus, $k[F]^U$ is a polynomial ring. It follows that F is normal; see, e.g., [29, Section 6]. \square

The base point $x \in X^0$ yields a base point $\tilde{x} \in \tilde{X}^0 = \hat{X}^0$, as follows: \hat{X}^0 is the total space of the direct sum of the line bundles on $X^0 = G/H$ associated with the invertible sheaves $\mathcal{O}_{G/H}(D)$, $D \in \mathcal{D}$, minus the union of the partial sums. Moreover, we may regard each f_D as a global section of $\mathcal{O}_{G/H}(D)$ which does not vanish at x (since $f_D(1) = 1$). Thus, there is a unique $\tilde{x} \in \hat{X}^0$ such that $f_D(\tilde{x}) = 1$ for all D .

Clearly, the isotropy group $\tilde{H} := \tilde{G}_{\tilde{x}}$ is the image of the homomorphism

$$H \rightarrow G \times \mathbb{G}_m^{\mathcal{D}}, \quad h \mapsto (h, (\chi_D(h), D \in \mathcal{D})).$$

The combinatorial data of the spherical homogeneous space \tilde{G}/\tilde{H} (that is, its weight group, colors, and valuation cone) are described in the following statement, which follows easily from Proposition 3.1.1.

Lemma 3.1.2.

- (1) The pairs $(\omega_D, \varepsilon_D)$, $D \in \mathcal{D}$, and the pairs $(0, \sum_D \langle D, \gamma_i \rangle, \varepsilon_D)$, $i = 1, \dots, r$, form a basis of the weight group $\tilde{\Lambda}(\tilde{X}) \subseteq \tilde{\Lambda} := \Lambda \times \mathbb{Z}^{\mathcal{D}}$. The dual basis of $\text{Hom}(\tilde{\Lambda}(\tilde{X}), \mathbb{Z})$ consists of the $\tilde{\rho}(v_{\tilde{D}})$, $D \in \mathcal{D}$, and $\tilde{v}_1, \dots, \tilde{v}_r$. The map $\tilde{\rho}: \tilde{\mathcal{D}} = \mathcal{D} \rightarrow \text{Hom}(\tilde{\Lambda}(\tilde{X}), \mathbb{Z})$ is injective.
- (2) The projection $p: \tilde{X} \rightarrow X$ yields a morphism $p^*: \Lambda(X) \rightarrow \tilde{\Lambda}(\tilde{X}) = \tilde{\Lambda}(\tilde{X})$ which sends any spherical root γ_i to

$$(\gamma_i, 0) = \sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle (\omega_D, \varepsilon_D) - \left(0, \sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle \varepsilon_D \right).$$

The $(\gamma_i, 0)$ and the $(\omega_D, \varepsilon_D)$, $D \in \mathcal{D}$, form another basis of $\text{Hom}(\tilde{\Lambda}(\tilde{X}), \mathbb{Z})$.

- (3) The transpose $p_*: \text{Hom}(\tilde{\Lambda}(\tilde{X}), \mathbb{R}) \rightarrow \text{Hom}(\Lambda(X), \mathbb{R})$ maps each $\tilde{\rho}(v_{\tilde{D}})$ to $\rho(v_D)$, and each \tilde{v}_i to v_i . Moreover, the valuation cone $\tilde{\mathcal{V}}$ is the preimage of \mathcal{V} under p_* .

Together with the classification of embeddings of spherical homogeneous spaces (see [21, Section 4]), this description implies readily the following:

Lemma 3.1.3. *The \tilde{G} -orbits in \tilde{X} correspond bijectively to those pairs (\mathcal{E}, I) , where $\mathcal{E} \subseteq \mathcal{D}$, $I \subseteq \{1, \dots, r\}$, such that the relative interior of $\text{Cone}(\rho(v_D))$, $D \in \mathcal{E}$; v_i , $i \in I$) meets the valuation cone \mathcal{V} . Equivalently, there exist positive integers x_D , $D \in \mathcal{E}$, such that $\sum_{D \in \mathcal{E}} x_D \langle D, \gamma_i \rangle \leq 0$ for all $i \notin I$.*

Under this correspondence $(\mathcal{E}, I) \leftrightarrow \mathcal{O}_{\mathcal{E}, I}$, the ideal of the closure $\bar{\mathcal{O}}_{\mathcal{E}, I}$ is generated as a G -module by the monomials $\prod_{i=1}^r s_i^{n_i} \prod_{D \in \mathcal{D}} s_D^{n_D}$, where $n_i > 0$ for all $i \in I$, and $n_D > 0$ for all $D \in \mathcal{E}$.

In particular, $q(\bar{\mathcal{O}}_{\mathcal{E}, I})$ is the linear subspace of \mathbb{A}^r defined by the vanishing of the s_i , $i \in I$. Moreover, the orbit $\mathcal{O}_{\mathcal{F}, J}$ is contained in $\bar{\mathcal{O}}_{\mathcal{E}, I}$ if and only if $\mathcal{E} \subseteq \mathcal{F}$ and $I \subseteq J$. The orbits contained in \tilde{X} correspond to the pairs (\emptyset, I) , where I is an arbitrary subset of $\{1, \dots, r\}$.

Example 3.1.4. If $X = \bar{G}_{\text{ad}}$, then \tilde{X} is nothing but the enveloping semigroup $\text{Env}(G)$ of Vinberg [39], as proved in [36, Theorem 20] (see also [37]; another proof will be given in Example 3.2.4). Moreover, the quotient morphism $q: \tilde{X} \rightarrow \mathbb{A}^r$ is the abelianization of $\text{Env}(G)$ defined in [39, p. 149]. Lemma 3.1.3 gives back Vinberg's description [39, Theorem 6] of the $G \times G$ -orbits in $\text{Env}(G)$, see [36, Section 5.3] for details.

3.2. An algebraic description

Consider the action of $G \times T$ on \tilde{X} , where G acts naturally and T acts via the homomorphism

$$u: T \rightarrow \mathbb{G}_m^{\mathcal{D}}, \quad t \mapsto (\omega_D(t))_{D \in \mathcal{D}}.$$

The open $G \times \mathbb{G}_m^{\mathcal{D}}$ -orbit \tilde{X}^0 is stable under this action; its structure is described by the following lemma.

Lemma 3.2.1. *There is an isomorphism of $G \times T$ -varieties*

$$\tilde{X}^0 \cong G/K \times^{T \cap H} T,$$

where K denotes the intersection of the kernels of all characters of H (this is a normal subgroup of H), and $T \cap H$ acts on G/K via the restriction of the right action of H on G/K .

Via this isomorphism, the restriction of the quotient map $q^0: \tilde{X}^0 \rightarrow \mathbb{A}^r$ is identified with the projection $G/K \times^{T \cap H} T \rightarrow T/T \cap H$, where $T/T \cap H$ is isomorphic to $\mathbb{G}_m^r \subset \mathbb{A}^r$ via the spherical roots $\gamma_1, \dots, \gamma_r$.

Proof. We have an isomorphism of $G \times \mathbb{G}_m^{\mathcal{D}}$ -varieties

$$\tilde{X}^0 \cong G \times^H \mathbb{G}_m^{\mathcal{D}},$$

where H acts on $\mathbb{G}_m^{\mathcal{D}}$ via its characters χ_D . This identifies q^0 to the H -invariant map

$$G \times \mathbb{G}_m^{\mathcal{D}} \rightarrow \mathbb{A}^r, \quad (g; t_D, D \in \mathcal{D}) \mapsto \left(\prod_{D \in \mathcal{D}} t_D^{\langle D, \gamma_i \rangle} \right)_{i=1, \dots, r}$$

with image \mathbb{G}_m^r . Since the subgroup $K \subseteq H$ acts trivially on $\mathbb{G}_m^{\mathcal{D}}$, we obtain

$$\tilde{X}^0 \cong G/K \times^{H/K} \mathbb{G}_m^{\mathcal{D}}.$$

Moreover, we have a commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & T \cap H & \longrightarrow & T & \longrightarrow & T/T \cap H \longrightarrow 1 \\
 & & \downarrow & & \downarrow u & & \downarrow \gamma_1, \dots, \gamma_r \\
 1 & \longrightarrow & H/K & \longrightarrow & \mathbb{G}_m^{\mathcal{D}} & \longrightarrow & \mathbb{G}_m^r \longrightarrow 1
 \end{array}$$

The square on the left is a fiber square, since all involved groups are diagonalizable and the dual square of character groups is

$$\begin{array}{ccc}
 \mathbb{Z}^{\mathcal{D}} & \longrightarrow & \mathcal{X}(H/K) = \mathcal{X}(H) \\
 \downarrow & & \downarrow \\
 \mathcal{X}(T) = \mathcal{X}(B) & \longrightarrow & \mathcal{X}(T \cap H) = \mathcal{X}(B \cap H)
 \end{array}$$

which is Cartesian by Lemma 2.1.1. It follows that the natural map

$$G/K \times^{T \cap H} T \rightarrow G/K \times^{H/K} \mathbb{G}_m^{\mathcal{D}}$$

is an isomorphism. \square

The subgroup K plays an important role in [28, Section 6]. As already observed, the quotient H/K is diagonalizable with character group $\mathcal{X}(H)$. Moreover, the homogeneous space G/K is quasi-affine and satisfies

$$k[G/K] = \bigoplus_{\chi \in \mathcal{X}(H)} k[G]_{\chi}^{(H)}.$$

In particular, the $G \times (H/K)$ -module $k[G/K]$ is multiplicity-free, and the algebra $k[G/K]^U = k[G]^{U \times K}$ is a polynomial ring in the f_D , $D \in \mathcal{D}$. It follows that the algebra $k[G/K]$ is finitely generated; see, e.g., [29, Corollary 4].

We now derive from Lemma 3.2.1 an algebraic description of the coordinate ring of the quasi-affine variety \tilde{X}^0 . For this, we introduce some notation: for any G -module M and any dominant weight λ , we denote by $M_{(\lambda)}$ the isotypical component of M of type λ , that is, the sum of all simple G -submodules with highest weight λ . Also, we denote by $(e^{\mu}, \mu \in \Lambda)$ the basis of the k -vector space $k[T]$ consisting of characters.

Lemma 3.2.2. *There is a isomorphism of $G \times T$ -algebras*

$$k[\tilde{X}^0] \cong \bigoplus_{\lambda \in \Lambda^+, \mu \in \Lambda, \lambda - \mu \in \Lambda(X)} k[G/K]_{(\lambda)} e^{\mu}, \quad (3.2.1)$$

where the right-hand side is a subalgebra of $k[G \times T] = \bigoplus_{\mu \in \Lambda} k[G] e^{\mu}$.

Proof. From Lemma 3.2.1 we obtain

$$k[\tilde{X}^0] \cong (k[G/K] \otimes k[T])^{T \cap H} \cong \bigoplus k[G]_{\chi}^{(H)} e^{\mu},$$

where the sum runs over those pairs $(\chi, \mu) \in \mathcal{X}(H) \times \Lambda$ such that $\chi|_{T \cap H} = \mu|_{T \cap H}$.

Moreover, given $\nu \in \mathcal{X}(T \cap H)$, the equality

$$\bigoplus_{\chi} k[G]_{\chi}^{(H)} = \bigoplus_{\lambda} k[G]_{(\lambda)}^{(H)}$$

holds, where the sum on the left-hand side (respectively right-hand side) runs over those $\chi \in \mathcal{X}(H)$ (respectively $\lambda \in \Lambda^+$) such that $\chi|_{T \cap H} = \nu$ (respectively $\lambda|_{T \cap H} = \nu$). Indeed, both sides are G -submodules of $k[G/K]$; and by Lemma 2.1.1, they have the same B -eigenvectors, namely, those monomials in the f_D , $D \in \mathcal{D}$, that have weight ν with respect to $T \cap H$.

Finally, the condition that $\lambda|_{T \cap H} = \mu|_{T \cap H}$ is equivalent to $\lambda - \mu \in \Lambda(X)$, since $\Lambda(X) = \mathcal{X}(T)^{T \cap H}$. Putting these facts together yields the desired isomorphism. \square

Next we deduce from Lemma 3.2.2 an algebraic description of the ring $R(X)$; to formulate it, we need an additional notation. Given $\lambda, \mu \in \Lambda$, we write

$$\lambda \leq_X \mu$$

if the difference $\mu - \lambda$ is a linear combination of spherical roots with non-negative coefficients; then $\mu - \lambda \in \Lambda(X)$, and $\lambda \leq \mu$ for the usual ordering on weights.

Theorem 3.2.3. *There is an isomorphism of $G \times T$ -algebras*

$$R(X) \cong \bigoplus_{\lambda \in \Lambda^+, \mu \in \Lambda, \lambda \leq_X \mu} k[G/K]_{(\lambda)} e^{\mu}, \quad (3.2.2)$$

where the left-hand side is a subalgebra of $k[G \times T]$. This isomorphism identifies each s_i with e^{γ_i} and each s_D with $f_D e^{\omega_D}$.

Proof. Via the isomorphism (3.2.1), the restriction $s_i|_{\tilde{X}^0}$ is identified with e^{γ_i} , since both span the subspace of $G \times T$ -eigenvectors of weight $(0, \gamma_i)$, and both take the value 1 at the base point \tilde{x} . Likewise, the restriction $s_D|_{\tilde{X}^0}$ is identified with $f_D e^{\omega_D}$. This implies (3.2.2), since $R(X)$ is the G -submodule of $k[\tilde{X}^0]$ generated by the monomials in s_1, \dots, s_r and the s_D , $D \in \mathcal{D}$. \square

Since $R(X)$ is a subalgebra of $k[G \times T]$, we must have

$$k[G/K]_{(\lambda)} k[G/K]_{(\mu)} \subseteq \bigoplus_{\nu \in \Lambda^+, \nu \leq_X \lambda + \mu} k[G/K]_{(\nu)}$$

for any dominant weights λ, μ , where the left-hand side denotes the product in $k[G/K]$. In other words, the *root monoid* of the algebra $k[G/K]$ (that is, the submonoid of Λ generated by the differences $\lambda + \mu - \nu$, where λ, μ, ν are dominant weights such that $k[G/K]_{(\lambda)} k[G/K]_{(\mu)}$ has a non-zero projection to $k[G/K]_{(\nu)}$) is contained in the monoid generated by the spherical roots.

In fact, both monoids generate the same cone in $\Lambda(X)_{\mathbb{R}}$ (see, e.g., [21, Lemma 6.1]; this result will be strengthened in Proposition 3.3.2). Thus, the subspaces

$$k[G/K]_{\leq_X \mu} := \bigoplus_{\lambda \in \Lambda, \lambda \leq_X \mu} k[G/K]_{(\lambda)}, \quad \mu \in \Lambda,$$

form an ascending filtration of $k[G/K]$ indexed by the partially ordered set Λ , and the right-hand side of (3.2.2) is the associated Rees algebra.

Theorem 3.2.3 yields a description of the fibers of the quotient map $q: \tilde{X} \rightarrow \mathbb{A}^r$ at all closed points. Indeed, let \tilde{X}_x be the fiber at $x = (x_1, \dots, x_r) \in \mathbb{A}^r$ and let $I = I(x)$ be the set of indices i such that $x_i \neq 0$. Then, by (3.2.2), the algebra $k[\tilde{X}_x] \cong R(X)/(s_1 - x_1, \dots, s_r - x_r)$ is isomorphic to $k[G/K]$ endowed with the new multiplication

$$m_I : k[G/K] \otimes k[G/K] \rightarrow k[G/K]$$

such that the restriction of m_I to any $k[G/K]_{(\lambda)} \otimes k[G/K]_{(\mu)}$ is the original multiplication

$$m : k[G/K]_{(\lambda)} \otimes k[G/K]_{(\mu)} \rightarrow \bigoplus_{\nu \in \Lambda^+, \nu \leq_X \lambda + \mu} k[G/K]_{(\nu)}$$

followed by the projection onto the partial sum of those $k[G/K]_{(\nu)}$ such that $\lambda + \mu - \nu$ is a linear combination of the $\gamma_i, i \in I$.

In particular, the general fibers $\tilde{X}_x, x \in \mathbb{G}_m^r$, satisfy $k[\tilde{X}_x] \cong k[G/K]$ as G -algebras. In other words, these fibers are all isomorphic to the *canonical embedding*

$$\text{CE}(G/K) := \text{Spec } k[G/K].$$

This is a normal affine G -variety in which the homogeneous space G/K is embedded as an open orbit with complement of codimension ≥ 2 (since G/K is quasi-affine and its coordinate ring is finitely generated).

On the other hand, in the coordinate ring of the special fiber \tilde{X}_0 , the product of any two simple G -submodules $V(\lambda), V(\mu)$ is just their Cartan product $V(\lambda + \mu)$. Thus, the G -variety \tilde{X}_0 is *horospherical*, i.e., the G -isotropy group of any point contains a maximal unipotent subgroup; see [29, Section 4]. In fact, the fibers of q realize a degeneration of $\text{CE}(G/K)$ to its “horospherical contraction” studied in [29]. Also, note that \tilde{X}_0 is the unique horospherical fiber.

Example 3.2.4. If $X = \tilde{G}_{\text{ad}}$, then we have to replace the acting group G with $G \times G$, and the subgroup K with $\text{diag}(G)$. Then the ring $k[G/K]$ is replaced with $k[(G \times G)/\text{diag}(G)] = k[G]$, which is isomorphic to $\bigoplus_{\lambda \in \Lambda} \text{End}(V(\lambda))$ as a $G \times G$ -module. It follows that

$$R(X) \cong \bigoplus_{\lambda \in \Lambda^+, \mu \in \Lambda, \lambda \leq \mu} \text{End}(V(\lambda)) e^{\mu},$$

where \leq is the usual ordering on weights. By [39, p. 152], this algebra is the coordinate ring of the enveloping monoid $\text{Env}(G)$. The horospherical degeneration of \tilde{X} is the asymptotic semigroup of [38]; the other fibers of the quotient morphism q are described in [3, Lemma 7.17].

By [39] again, the unit group of $\text{Env}(G)$ is $G \times^Z T$, where the center Z acts diagonally on $G \times T$. Alternatively, this follows from Lemma 3.2.1, which also implies that the action of $G \times T$ on $R(X)$ (where X is any wonderful G -variety) factors through an action of $G \times^Z T$. In fact, the latter action extends to an action of the monoid $\text{Env}(G)$ (as follows from Theorem 3.2.3) which may deserve a detailed investigation.

3.3. Generators and relations

For any $D \in \mathcal{D}$, let $V_D \cong V(\omega_D)$ be the G -submodule of $\Gamma(X, \mathcal{O}_X(D))$ generated by s_D . Proposition 3.1.1 implies readily that the algebra $R(X)$ is generated by s_1, \dots, s_r and the G -modules V_D , where $D \in \mathcal{D}$. Thus, we may write $R(X) = S/I$, where I is an ideal of

$$S := k[s_1, \dots, s_r] \otimes \text{Sym}\left(\bigoplus_{D \in \mathcal{D}} V_D\right).$$

To construct generators of I , we determine the product in $R(X)$ of any two submodules V_{D_1}, V_{D_2} , as a quotient of their tensor product $V_{D_1} \otimes V_{D_2}$.

Regard each V_D as the submodule of $k[G/K]$ generated by f_D . More generally, given a family of non-negative integers $(n_D)_{D \in \mathcal{D}}$, let $V_{\sum n_D D}$ be the G -submodule of $k[G/K]$ generated by $\prod f_D^{n_D}$. Then

$$k[G/K] = \bigoplus_{(n_D)} V_{\sum n_D D},$$

where the sum runs over all families of non-negative integers. Thus, any product $V_{D_1} V_{D_2}$ in $k[G/K]$ may be decomposed into a partial sum of $V_{\sum n_D D}$'s, where

$$\sum_{D \in \mathcal{D}} n_D \omega_D \leq_X \omega_{D_1} + \omega_{D_2} \quad \text{and} \quad \sum_{D \in \mathcal{D}} n_D \chi_D = \chi_{D_1} + \chi_{D_2}.$$

In particular, we may index the components $V_{\sum n_D D}$ of $V_{D_1} V_{D_2}$ by their highest weight $\lambda = \sum_{D \in \mathcal{D}} n_D \omega_D$. Moreover, each component $V(\lambda)$ may be embedded into the tensor product $\bigotimes_{D \in \mathcal{D}} V_D^{\otimes n_D}$ as its Cartan component. This yields morphisms of G -modules

$$p_{D_1, D_2}^\lambda : V_{D_1} \otimes V_{D_2} \rightarrow \text{Sym}\left(\bigoplus_{D \in \mathcal{D}} V_D\right),$$

obtained as the composition

$$V_{D_1} \otimes V_{D_2} \rightarrow V_{D_1} V_{D_2} \rightarrow V(\lambda) \rightarrow \bigotimes_{D \in \mathcal{D}} V_D^{\otimes n_D} \rightarrow \text{Sym}\left(\bigoplus_{D \in \mathcal{D}} V_D\right).$$

Note that each p_{D_1, D_2}^λ is only defined up to multiplication by a non-zero scalar, since the same holds for the projection $V_{D_1} V_{D_2} \rightarrow V(\lambda)$.

Proposition 3.3.1. *With the above notation, there exist unique normalizations of the morphisms p_{D_1, D_2}^λ such that the ideal I is generated by the elements*

$$v_1 \otimes v_2 - \sum_{\lambda \in \Lambda^+, \lambda \leq_X \omega_{D_1} + \omega_{D_2}} p_{D_1, D_2}^\lambda(v_1 \otimes v_2) \prod_{i=1}^r s_i^{n_i}, \quad (3.3.1)$$

where $v_1 \in V_{D_1}$, $v_2 \in V_{D_2}$, and n_1, \dots, n_r are the non-negative integers defined by $\omega_{D_1} + \omega_{D_2} - \lambda = \sum_{i=1}^r n_i \gamma_i$.

Proof. It follows from Theorem 3.2.3 that I contains the elements (3.3.1) for suitable normalizations of the p_{D_1, D_2}^λ . Let $J \subseteq I$ be the ideal of S generated by all these elements. The natural map $S \rightarrow R(X)$ factors through a surjective homomorphism of G -algebras

$$h: S/J \rightarrow R(X).$$

We claim that the induced (surjective) homomorphism

$$\bar{h}: S/(J, s_1, \dots, s_r) \rightarrow R(X)/(s_1, \dots, s_r)$$

is an isomorphism. Indeed, the left-hand side is generated as a G -algebra by simple modules V_D , $D \in \mathcal{D}$, such that the product of any two such modules is their Cartan product, or zero. By a result of Kostant (see, e.g., [8, 4.1 Lemme]), it follows that the product of any two simple modules in this G -algebra is again their Cartan product, or zero. Thus, the subalgebra

$$(S/(J, s_1, \dots, s_r))^U = S^U/(J^U, s_1, \dots, s_r)$$

is generated by the images of the U -invariant lines in the V_D . On the other hand,

$$(R(X)/(s_1, \dots, s_r))^U = R(X)^U/(s_1, \dots, s_r)$$

is a polynomial ring in the images of the s_D , $D \in \mathcal{D}$, by Proposition 3.1.1. This implies our claim.

From this claim and the freeness of the $k[s_1, \dots, s_r]$ -module $R(X)$ (Proposition 3.1.1 again), it follows that

$$\ker(h) = (s_1, \dots, s_r) \ker(h).$$

On the other hand, the algebra S is graded by the weight group Λ , where each s_i has degree γ_i and each s_D has degree ω_D . Note that the degrees of non-zero homogeneous elements of S are linear combinations of simple roots with non-negative coefficients, and that the homogeneous elements of degree 0 are just the constants. Hence S admits a coarser positive grading. Moreover, the generators of J are all homogeneous, and the ideal of $R(X)$ is homogeneous as well (the induced grading on $R(X)$ arises from the morphism $u: T \rightarrow \mathbb{G}_m^{\mathcal{D}}$, $t \mapsto (\omega_D(t))$). Thus, $\ker(h)$ is also graded. By the graded Nakayama lemma, it follows that $\ker(h) = 0$. \square

Proposition 3.3.2. *For any spherical root γ_i , there exist colors D_1, D_2 and a dominant weight λ such that the G -module $V(\lambda)$ occurs in the product $V_{D_1} V_{D_2} \subset k[G/K]$, and $\omega_{D_1} + \omega_{D_2} - \lambda$ is a positive integral multiple of γ_i .*

Proof. Let J be the ideal of the polynomial ring $k[s_1, \dots, s_r]$ generated by those monomials $\prod_{i=1}^r s_i^{n_i}$ that appear in some relation (3.3.1). Then the quotient $R(X)/JR(X)$ is a G -algebra generated by the images of the trivial G -modules s_1, \dots, s_r and of the G -modules V_D , $D \in \mathcal{D}$, and the product of any two such modules is their Cartan product. As in the proof of Proposition 3.3.1, it follows that the product of any two simple modules in the G -algebra $R(X)/JR(X)$ is their Cartan product, or zero. So, for any point x of the zero set $Z(J)$ in \mathbb{A}^r , the fiber \tilde{X}_x is horospherical. Thus, $Z(J)$ is just the origin, and hence J contains positive powers of s_1, \dots, s_r . \square

Example 3.3.3. The algebra $R(\tilde{G}_{\text{ad}})$ is generated by s_1, \dots, s_r and the $G \times G$ -submodules $\text{End}(V(\omega_1)), \dots, \text{End}(V(\omega_r))$. For $1 \leq i \leq j \leq r$, we have a canonical isomorphism

$$V(\omega_i) \otimes V(\omega_j) \cong \bigoplus_{\lambda \in \Lambda^+} N_{ij}^\lambda \otimes V(\lambda),$$

where $N_{ij}^\lambda := \text{Hom}^G(V(\lambda), V(\omega_i) \otimes V(\omega_j))$. Thus, the tensor product $\text{End}(V(\omega_i)) \otimes \text{End}(V(\omega_j))$ is the direct sum of the $\text{End}(N_{ij}^\lambda) \otimes \text{End}(V(\lambda))$. Moreover, the product $\text{End}(V(\omega_i)) \text{End}(V(\omega_j))$ in $k[G]$ is obtained by projecting each $\text{End}(N_{ij}^\lambda)$ to k via the trace map. This yields the relations in $R(\tilde{G}_{\text{ad}})$:

$$\text{End}(V(\omega_i)) \text{End}(V(\omega_j)) \cong \bigoplus_{\lambda} \text{End}(V(\lambda)) \prod_{i=1}^r s_i^{n_i},$$

where the sum runs over those dominant weights λ such that $V(\lambda)$ occurs in $V(\omega_i) \otimes V(\omega_j)$, and $\omega_i + \omega_j - \lambda = \sum_{i=1}^r n_i \alpha_i$. Since the G -module $V(2\omega_i - \alpha_i)$ occurs in $V(\omega_i) \otimes V(\omega_i)$, the root monoid is generated by the simple roots $\alpha_1, \dots, \alpha_r$.

In particular, if $G = \text{SL}_n$ then $V(\omega_i) = \bigwedge^i k^n$ for $i = 1, \dots, n-1$, and

$$V(\omega_i) \otimes V(\omega_j) \cong \bigoplus_{i', j', 0 \leq i' \leq i \leq j \leq j' \leq n} V(\omega_{i'} + \omega_{j'}),$$

where we set $\omega_0 = \omega_n = 0$. Thus, the relations have degree at most 2 in the $\text{End}(V(\omega_i))$. This also holds for $G = \text{Sp}_{2n}$, but not for any other simple group.

4. The total coordinate ring of a spherical variety

4.1. The equivariant divisor class group

Let X be a spherical variety under the connected reductive group G . Replacing G with a finite cover, we may assume that $G = \mathbf{G} \times C$, where C is a torus (the connected center of G) and \mathbf{G} is a simply-connected semi-simple group (the derived subgroup of G). Then $k[G]$ is a UFD, and $k[G]^* = k[C]^*$ consists of the scalar multiples of characters of C (identified with characters of G). Moreover, replacing C with a quotient torus, we may assume that C acts faithfully on X .

We denote by $X^0 \subseteq X$ the open G -orbit and by X_1, \dots, X_n the irreducible components of codimension 1 of the boundary $X \setminus X^0$; they form the (possibly empty) set of *boundary divisors*. Let $X^{\leq 1}$ be the union of X^0 and all G -orbits of codimension 1, i.e., of the open G -orbits in

the boundary divisors. Clearly, $X^{\leq 1}$ is an open G -stable subset of X , and its complement has codimension ≥ 2 . Moreover, since X is a normal G -variety, the open subset $X^{\leq 1}$ is non-singular, and $k[X] = k[X^{\leq 1}]$. We shall assume for simplicity that $k[X] = k$. This holds, e.g., if X is complete.

Recall that a *divisorial sheaf* on X is a coherent sheaf \mathcal{F} which is reflexive of rank 1; equivalently, $\mathcal{F} \cong \mathcal{O}_X(D)$ for some Weil divisor D on X . Then \mathcal{F} is canonically isomorphic to $j_*(j^*\mathcal{F})$, where

$$j : X^{\leq 1} \rightarrow X$$

denotes the inclusion; moreover, the sheaf $j^*\mathcal{F}$ is invertible. Conversely, any invertible sheaf \mathcal{L} on $X^{\leq 1}$ defines a divisorial sheaf $\mathcal{F} := j_*\mathcal{L}$, and \mathcal{L} may be identified with $j^*\mathcal{F}$. So we shall freely identify divisorial sheaves on X with invertible sheaves on $X^{\leq 1}$. In particular, the tensor product of invertible sheaves defines the product of divisorial sheaves (the double dual of their tensor product).

Together with [25, Section 2.4], it follows that any divisorial sheaf has G -linearizations, and any two of them differ by a unique character of C . The group of isomorphism classes of G -linearized divisorial sheaves on X may be identified with $\text{Pic}^G(X^{\leq 1})$; we denote this group by $\text{Cl}^G(X)$ and call it the *equivariant divisor class group*. (This is the equivariant Chow group $A_{\dim(X)-1}^G(X)$ of [17].) Note the exact sequence

$$0 \rightarrow \mathcal{X}(C) \rightarrow \text{Cl}^G(X) \rightarrow \text{Cl}(X) \rightarrow 0, \quad (4.1.1)$$

where $\text{Cl}(X)$ denotes the usual class group.

We shall obtain a presentation of the abelian group $\text{Cl}^G(X)$. For this, we generalize some of the notation of 2.1: we define the open B -orbit X_B^0 , the set \mathcal{D} of irreducible components of $X^0 \setminus X_B^0$ (these are prime B -stable divisors that we also regard as divisors in X), the weight group $\Lambda(X) \cong k(X)^{(B)}/k^*$, the valuations v_{X_i} , v_D of the function field $k(X)$ associated with the boundary divisors and the colors, and their restrictions v_i , $\rho(v_D)$ to $\text{Hom}(\Lambda(X), \mathbb{R})$. We choose a base point $x \in X_B^0$ and denote by H its stabilizer in G .

The pull-back of each $D \in \mathcal{D}$ under the projection $\pi : G \rightarrow G/H \cong X^0$ is a $B \times H$ -stable divisor, which has a unique equation $f_D \in k[G]$ such that $f_D(1) = 1$ and f_D is invariant under left multiplication by C . Then f_D is a $B \times H$ -eigenvector; let (ω_D, χ_D) be its weight. Also, let $V_D \subset k[G]$ be the span of the left G -translates of f_D . The G -module V_D is isomorphic to $V(\omega_D)$ and consists of right H -eigenvectors of weight χ_D . It yields a G -morphism $G/H \rightarrow \mathbb{P}(V_D)^*$ which extends to a G -morphism

$$\varphi : X^{\leq 1} \rightarrow \mathbb{P}(V_D)^*.$$

Moreover, D is the pull-back under φ of the hyperplane of $\mathbb{P}(V_D)^*$ orthogonal to $f_D \in V_D$. This yields a linearization of the invertible sheaf $\mathcal{O}_{X^{\leq 1}}(D) \cong \varphi^*\mathcal{O}_{\mathbb{P}(V_D)^*}(1)$, and hence a *canonical* linearization of the divisorial sheaf $\mathcal{O}_X(D)$. We denote by $[D]^G \in \text{Cl}^G(X)$ the corresponding class (notice, however, that D is never G -stable). We also obtain classes $[X_i]^G$ associated with boundary divisors, since the sheaves $\mathcal{O}_X(X_i)$ are canonically G -linearized.

Proposition 4.1.1.

(1) The abelian group $\mathrm{Cl}^G(X)$ is defined by generators

$$[X_1]^G, \quad \dots, \quad [X_n]^G, \quad [D]^G \quad (D \in \mathcal{D}), \quad \chi \in \mathcal{X}(C)$$

and relations

$$\sum_{i=1}^n v_i(\lambda)[X_i]^G + \sum_{D \in \mathcal{D}} \rho(v_D)(\lambda)[D]^G = \lambda|_C \quad (4.1.2)$$

for all $\lambda \in \Lambda(X) \subseteq \mathcal{X}(T)$.

(2) Likewise, the abelian group $\mathrm{Cl}(X)$ is defined by generators $[X_1], \dots, [X_n], [D] \quad (D \in \mathcal{D})$, and relations

$$\sum_{i=1}^n v_i(\lambda)[X_i] + \sum_{D \in \mathcal{D}} \rho(v_D)(\lambda)[D] = 0$$

for all $\lambda \in \Lambda(X)$.

Proof. We begin with the easy proof of (2). Since X_B^0 has trivial divisor class group, the group $\mathrm{Cl}(X)$ is generated by the classes of boundary divisors and of colors. Moreover, all relations between these B -stable divisors arise from divisors of rational functions on X which are B -eigenvectors.

Together with (4.1.1), it follows that $\mathrm{Cl}^G(X)$ is generated by the above classes and the characters of C . Moreover, any $f \in k(X)^{(B)}$ may be regarded as a C -invariant rational section of the trivial line bundle on X with linearization twisted by $-\lambda|_C$, where λ denotes the weight of f . Thus, $\mathrm{div}(f) - \lambda|_C = 0$ in $\mathrm{Cl}^C(X) \cong \mathrm{Cl}^G(X)$. In other words, the relations (4.1.2) hold in $\mathrm{Cl}^G(X)$.

Assume that

$$\sum_{i=1}^n x_i [X_i]^G + \sum_{D \in \mathcal{D}} y_D [D]^G = \chi$$

in $\mathrm{Cl}^G(X)$, where x_i, y_D are integers and $\chi \in \mathcal{X}(C)$. Then

$$\sum_{i=1}^n x_i [X_i] + \sum_{D \in \mathcal{D}} y_D [D] = 0$$

in $\mathrm{Cl}(X)$. Thus, there exists $\lambda \in \Lambda(X)$ such that $v_i(\lambda) = x_i$ for all i , and $\rho(v_D)(\lambda) = y_D$ for all D . Hence $\lambda|_C = \chi$ in $\mathrm{Cl}^G(X)$. By (4.1.1), $\lambda|_C = \chi$ in $\mathcal{X}(C)$, so that (4.1.2) gives all the relations. \square

As in 2.2, the inclusion

$$i: G/H \rightarrow X \quad (4.1.3)$$

defines the restriction map

$$i^*: \mathrm{Cl}^G(X) = \mathrm{Pic}^G(X^{\leq 1}) \rightarrow \mathrm{Pic}^G(G/H) \cong \mathcal{X}(H),$$

where the latter isomorphism is defined by $\chi \mapsto \mathcal{L}(\chi)$. Moreover, the first assertion of Proposition 2.2.1 may be easily generalized as follows.

Lemma 4.1.2.

(1) *The sequence*

$$0 \rightarrow \mathbb{Z}^n \xrightarrow{u} \mathrm{Cl}^G(X) \xrightarrow{i^*} \mathrm{Pic}^G(G/H) \rightarrow 0$$

is exact, where $u(x_1, \dots, x_n) = \sum_{i=1}^n x_i [X_i]^G$.

(2) *The canonical section $s_D \in \Gamma(X, \mathcal{O}_X(D))$ is a B -eigenvector of weight ω_D ; the canonical section $s_i \in \Gamma(X, \mathcal{O}_X(X_i))$ is G -invariant.*

Example 4.1.3. In the case where X is a toric variety (i.e., $G = T = C$ is a torus), the group $\mathrm{Cl}^T(X)$ is freely generated by the classes of boundary divisors, as follows, e.g., from Lemma 4.1.2. In particular, this group is torsion-free, while the torsion subgroup of $\mathrm{Cl}(X)$ may be any prescribed finite abelian group.

For an arbitrary spherical G -variety X , the group $\mathrm{Cl}^G(X)$ may have non-zero torsion. Specifically, consider the standard embedding X of the group SO_{2n} , i.e., the normalization of the wonderful compactification of the semi-simple adjoint group PSO_{2n} in the function field of SO_{2n} . Then $G = \mathbf{G} = \mathrm{Spin}_{2n} \times \mathrm{Spin}_{2n}$, so that $\mathrm{Cl}^G(X) = \mathrm{Cl}(X)$. Using Proposition 4.1.1, one checks that this group is the direct product of its subgroup $\mathrm{Pic}(X)$ (freely generated by the classes of the n colors) and a group of order 2.

Some of the results of 2.3 also extend to spherical varieties. Specifically, let $\mathrm{Eff}^G(X) \subseteq \mathrm{Cl}^G(X)$ be the submonoid consisting of classes of sheaves having non-zero global sections. Then $\mathrm{Eff}^G(X)$ is the pull-back under the map $\mathrm{Cl}^G(X) \rightarrow \mathrm{Cl}(X)$ of the effective monoid $\mathrm{Eff}(X) \subseteq \mathrm{Cl}(X)$, and the latter is generated by the classes of boundary divisors and of colors. Moreover, the associated cone $\mathrm{Eff}(X)_{\mathbb{R}} \subset \mathrm{Cl}(X)_{\mathbb{R}}$ is a rational polyhedral cone, which contains no line since $k[X] = k$.

The extremal rays of $\mathrm{Eff}(X)_{\mathbb{R}}$ which contain no color are still the rays generated by the fixed boundary divisors, as in Lemma 2.3.2. Moreover, the extremal rays containing no boundary divisor correspond to G -morphisms $X^{\leq 1} \rightarrow G/P$ where $P \supseteq H$ is a maximal parabolic subgroup of G , as in Lemma 2.3.3. But the statement of Lemma 2.3.1 does not hold in general, since two boundary divisors may well have the same class (e.g., if $G = \mathbb{G}_m$ and $X = \mathbb{P}^1$). Likewise, Lemma 2.3.4 admits no straightforward generalization, as shown by the example of the projective cone over \mathbb{P}^1 embedded via the sections of $\mathcal{O}(n)$.

We shall determine the extremal rays of $\mathrm{Eff}(X)_{\mathbb{R}}$ when X is the *standard embedding* of a *sober* spherical homogeneous space G/H , i.e., the group $N_G(H)/H$ is finite, and X is the unique complete toroidal embedding with only one closed orbit. Note that H contains C , so that we may assume G to be semi-simple. The variety X is generally singular, but with quotient singularities only; it admits a finite surjective G -morphism

$$f: X \rightarrow \mathbf{X},$$

where \mathbf{X} is a wonderful G -variety, and the preimage of any color of \mathbf{X} is a unique color (see [24, Corollary 7.6]; in fact, there is a canonical choice of \mathbf{X} , see [28] or Section 4.3). Moreover, the Picard group of X is still freely generated by the classes of colors, see [10]. Thus, $f^*: \text{Pic}(\mathbf{X}) \rightarrow \text{Pic}(X)$ yields isomorphisms

$$\text{Pic}(\mathbf{X})_{\mathbb{R}} \cong \text{Pic}(X)_{\mathbb{R}} = \text{Cl}(X)_{\mathbb{R}} = \text{Cl}^G(X)_{\mathbb{R}}, \quad \text{Eff}(\mathbf{X})_{\mathbb{R}} \cong \text{Eff}(X)_{\mathbb{R}}.$$

Recall from Theorem 2.4.1 that the extremal rays of $\text{Eff}(\mathbf{X})_{\mathbb{R}}$ are generated by the classes of fixed boundary divisors, and of colors associated with surjective morphisms $\mathbf{X} \rightarrow \mathbf{X}'$ with connected fibers, where \mathbf{X}' is a wonderful G -variety of rank ≤ 1 having a unique color. Moreover, f induces a bijection between the boundary divisors $\mathbf{X}_1, \dots, \mathbf{X}_r$ and X_1, \dots, X_r , which restricts to a bijection between the subsets of fixed divisors (since f is finite). Likewise, the morphisms $\mathbf{X} \rightarrow \mathbf{X}'$ as above are in bijection with morphisms $X \rightarrow X'$ satisfying the same assumptions. This yields:

Proposition 4.1.4. *Let X be the standard embedding of a sober spherical homogeneous space. Then the extremal rays of $\text{Eff}(X)_{\mathbb{R}}$ consist of the rays generated by fixed boundary divisors, and of those generated by pull-backs of colors under G -morphisms $\varphi: X \rightarrow X'$, where X' is a wonderful G -variety of rank 1 having a unique color (in particular, X' is homogeneous). Both types of rays are disjoint.*

Also, note that a boundary divisor is either fixed or semi-ample (i.e., some positive multiple is base-point-free). However, a semi-ample boundary divisor X_i may well satisfy $\Gamma(X, \mathcal{O}_X(X_i)) = ks_i$ and hence be stable under $\text{Aut}^0(X)$. This happens, e.g., for the two boundary divisors of the standard embedding of the group SO_4 .

4.2. The total coordinate ring

Our first aim is to define this ring $R(X)$, where X is a spherical variety under the connected reductive group G . The idea is to choose G -linearized divisorial sheaves $\mathcal{F}(\delta)$ in all the classes $\delta \in \text{Cl}^G(X)$, and to endow the sum

$$\bigoplus_{\delta \in \text{Cl}^G(X)} \Gamma(X, \mathcal{F}(\delta))$$

with a multiplication law. For this, we need morphisms

$$\mathcal{F}(\delta) \otimes \mathcal{F}(\delta') \rightarrow \mathcal{F}(\delta + \delta')$$

so that the resulting law is commutative and associative. The existence of such representatives and morphisms is guaranteed by the following:

Lemma 4.2.1. *Every $\delta \in \text{Cl}^G(X)$ is the class of a unique G -linearized divisorial subsheaf*

$$\mathcal{F}(\delta) \subseteq i_* \mathcal{L}(\chi),$$

where i is defined by (4.1.3), and $\chi = i^*(\delta) \in \mathcal{X}(H)$. In particular, $\Gamma(X, \mathcal{F}(\delta))$ is a G -submodule of $k[G]_{\chi}^{(H)}$ which depends only on δ .

Moreover, for any $\delta' \in \mathrm{Cl}^G(X)$ with associated character $\chi' \in \mathcal{X}(H)$, the canonical homomorphism $i_*\mathcal{L}(\chi) \otimes i_*\mathcal{L}(\chi') \rightarrow i_*\mathcal{L}(\chi + \chi')$ maps $\mathcal{F}(\delta) \otimes \mathcal{F}(\delta')$ to $\mathcal{F}(\delta + \delta')$.

Proof. Let \mathcal{F} be a representative of δ . Then the natural map $\mathcal{F} \rightarrow i_*(i^*\mathcal{F})$ identifies \mathcal{F} with a subsheaf of $i_*\mathcal{L}(\chi)$. To show that this subsheaf only depends on δ , we may assume that $X = X^{\leq 1}$. Then X is covered by G -stable open subsets consisting of two G -orbits: the open orbit and an orbit of codimension 1. So we may further assume that X is one of these “elementary embeddings” of G/H .

Let $P \supseteq B$ be the stabilizer in G of the open B -orbit X_B^0 , and let $X_B \subseteq X$ be the complement of the union of the colors. Then X_B is an affine P -stable open subset of X , isomorphic to $P \times^L \mathbb{A}^1$ where L is a Levi subgroup of P , acting on \mathbb{A}^1 via a character λ ; this induces a P -equivariant isomorphism $X_B^0 \cong P \times^L \mathbb{G}_m$ (see, e.g., [10, Section 1.1]). Since X is covered by the G -translates of X_B , it suffices to check that the image of $\Gamma(X_B, \mathcal{F})$ in $\Gamma(X_B^0, \mathcal{L}(\chi))$ depends only on δ .

Using the correspondence between P -linearized sheaves on $P \times^L \mathbb{A}^1$ and L -linearized sheaves on \mathbb{A}^1 , we may assume that $X = \mathbb{A}^1$ where $G = L$ acts via $\lambda \in \mathcal{X}(C)$. Then \mathcal{F} is isomorphic to \mathcal{O}_X with linearization twisted by $\chi \in \mathcal{X}(C)$, and δ may be identified with χ . Moreover, $\Gamma(X_B^0, \mathcal{L}(\chi))$ is the subspace of $k[C]$ spanned by the characters $\chi + n\lambda$ where $n \in \mathbb{Z}$, while the image of $\Gamma(X, \mathcal{F})$ is spanned by the $\chi + n\lambda$ where n is non-negative. This completes the proof of the first assertion; the second one may be checked similarly. \square

As a straightforward consequence, we obtain the following proposition.

Proposition 4.2.2. *With the notation of Lemma 4.2.1, the sum*

$$\bigoplus_{\delta \in \mathrm{Cl}^G(X)} \mathcal{F}(\delta)$$

has a natural structure of a sheaf of \mathcal{O}_X -algebras, graded by $\mathrm{Cl}^G(X)$, and compatibly G -linearized. In particular,

$$R(X) := \bigoplus_{\delta \in \mathrm{Cl}^G(X)} \Gamma(X, \mathcal{F}(\delta))$$

has a natural structure of a G -algebra, graded by the abelian group $\mathrm{Cl}^G(X)$, and depending only on $X^{\leq 1}$.

Next, recall that $\mathcal{X}(C)$ is a subgroup of $\mathrm{Cl}^G(X)$, and the corresponding G -linearized sheaves are just copies of the structure sheaf \mathcal{O}_X with linearization twisted by characters. Since $\Gamma(X, \mathcal{O}_X) = k$, the subalgebra

$$\bigoplus_{\delta \in \mathcal{X}(C)} \Gamma(X, \mathcal{F}(\delta)) \subseteq R(X)$$

is isomorphic to the coordinate ring $k[C]$. Together with (4.1.1), it follows that $R(X)^* \cong k[C]^* = k^*\mathcal{X}(C)$.

By the same arguments as in Proposition 3.1.1, we obtain:

Proposition 4.2.3. *The invariant algebra $R(X)^U$ is freely generated as a $k[C]$ -algebra by s_1, \dots, s_n and the s_D , $D \in \mathcal{D}$.*

In particular, the subalgebra $R(X)^G$ (respectively $R(X)^G$) is freely generated by s_1, \dots, s_n as a $k[C]$ -algebra (respectively as a k -algebra).

Together with [29], it follows that the k -algebra $R(X)$ is finitely generated and normal. Thus, it corresponds to a normal affine variety \tilde{X} equipped with an action of the group

$$\tilde{G} := G \times \Gamma_X,$$

where we put

$$\Gamma_X := \text{Hom}(\text{Cl}^G(X), \mathbf{G}_m),$$

the diagonalizable group with character group $\text{Cl}^G(X)$. The sheaf of \mathcal{O}_X -algebras

$$\bigoplus_{\delta \in \text{Cl}^G(X)} \mathcal{F}(\delta)$$

yields a \tilde{G} -variety equipped with an affine morphism $p: \hat{X} \rightarrow X$ which is G -equivariant and Γ_X -invariant. Moreover, p is a good quotient for the action of Γ_X (that is, the natural map $\mathcal{O}_X \rightarrow p_*(\mathcal{O}_{\hat{X}})^{\Gamma_X}$ is an isomorphism), and the restriction of p to the open subset $X^{\leq 1}$ (or to the regular locus of X) is a principal bundle for that group.

As in [5] or in 3.1, one checks that the natural map $\iota: \hat{X} \rightarrow \tilde{X}$ is an open immersion, the complement of its image has codimension ≥ 2 , and $R(X)$ is a UFD. Moreover, \hat{X} contains an open orbit of a Borel subgroup of the connected reductive group \tilde{G}^0 , so that the \tilde{G}^0 -variety \tilde{X} is spherical.

One may show, as in Proposition 3.1.1, that the quotient map $\tilde{X} \rightarrow \tilde{X}/G$ is flat, and its fibers are normal varieties. This also follows from a structure theorem for $R(X)$ that we shall obtain in the next subsection.

Example 4.2.4. In the toric case, Proposition 4.2.3 yields an algebra isomorphism $R(X) \cong k[T][s_1, \dots, s_n]$; equivalently, $\tilde{X} \cong T \times \mathbb{A}^n$. Moreover, $\tilde{G} \cong T \times \mathbb{G}_m^n$, where T (respectively \mathbb{G}_m^n) acts on \tilde{X} via its natural action on the first (respectively second) factor. One may check that the complement $\tilde{X} \setminus \hat{X}$ is the union of the subspaces $(x_i = 0)$, $i \in I$, where the x_i are the coordinates on \mathbb{A}^n , and $I \subseteq \{1, \dots, n\}$ is such that $\bigcap_{i \in I} X_i$ is empty.

4.3. Structure

We begin by recalling how to associate a wonderful variety to any spherical variety, after [28]. The notation is as in 4.1; in particular, X denotes a spherical variety under $G = \mathbf{G} \times C$, with open orbit G/H .

The normalizer $N_G(H)$ acts on G/H by equivariant automorphisms, and permutes the colors $D \in \mathcal{D}$. The subgroup of $N_G(H)$ which stabilizes all colors is called the *spherical closure* of H in [28]. Clearly, this subgroup has finite index in $N_G(H)$, and contains both groups H and C ;

thus, it may be written as $\mathbf{H} \times C$, where \mathbf{H} is a closed subgroup of the derived subgroup \mathbf{G} . The homogeneous space \mathbf{G}/\mathbf{H} is spherical and sober; its set of colors may be identified with \mathcal{D} .

We record the following observation, implicit in [28, Section 6].

Lemma 4.3.1. *Let K be the intersection of the kernels of all characters of H . Then K is also the intersection of the kernels of all characters of \mathbf{H} .*

Proof. Denote the latter group by \mathbf{K} , we may identify this group with the intersection of all characters of $\mathbf{H} \times C$. Then $\mathbf{K} \subseteq H$ and H/\mathbf{K} is diagonalizable, since the quotient $(\mathbf{H} \times C)/H \subseteq N_G(H)/H$ is diagonalizable. Thus, $K \subseteq \mathbf{K} \subseteq H$.

The group $\mathcal{X}(H)$ is generated by the restrictions of characters of C and the weights of the f_D , $D \in \mathcal{D}$ (see, e.g., the proof of Lemma 6.3.1 in [28]). Thus, each character of H extends to a character of $\mathbf{H} \times C$. It follows that $\mathbf{K} \subseteq K$. \square

By [24, Corollary 7.6], the homogeneous space \mathbf{G}/\mathbf{H} admits a unique wonderful embedding \mathbf{X} . The natural map

$$G/H \rightarrow G/(\mathbf{H} \times C) = \mathbf{G}/\mathbf{H}$$

extends uniquely to a G -morphism

$$\varphi: X^{\leq 1} \rightarrow \mathbf{X}$$

which is C -invariant; moreover, φ extends to the whole variety X if and only if this variety is toroidal.

Recall that each invertible sheaf \mathcal{L} on \mathbf{X} carries a canonical \mathbf{G} -linearization; thus, its pull-back $\varphi^*\mathcal{L}$ is G -linearized, where C acts trivially. This defines a homomorphism

$$\varphi^*: \text{Pic}(\mathbf{X}) = \text{Pic}^G(\mathbf{X}) \rightarrow \text{Pic}^G(X^{\leq 1}) = \text{Cl}^G(X) \quad (4.3.1)$$

and morphisms of G -modules

$$\varphi^*: \Gamma(\mathbf{X}, \mathcal{L}) \rightarrow \Gamma(X^{\leq 1}, \varphi^*\mathcal{L})$$

(via the natural map $\mathcal{L} \rightarrow \varphi_*(\varphi^*\mathcal{L}) = \mathcal{L} \otimes \varphi_*\mathcal{O}_{X^{\leq 1}}$). Note that

$$\varphi^*([\mathbf{D}]) = [D]^G \quad (4.3.2)$$

for each $D \in \mathcal{D}$, where \mathbf{D} denotes the corresponding color of \mathbf{X} .

In turn, we obtain a homomorphism of graded G -algebras

$$\varphi^*: R(\mathbf{X}) \rightarrow R(X),$$

where the corresponding homomorphism between their grading groups is given by (4.3.1).

Theorem 4.3.2.(1) *The restriction*

$$\varphi^*G : R(\mathbf{X})^G = k[s_1, \dots, s_r] \rightarrow R(X)^G = k[C][s_1, \dots, s_n]$$

is given by

$$\varphi^*G(\mathbf{s}_i) = \prod_{j=1}^n s_j^{-v_j(\gamma_i)}$$

for $i = 1, \dots, r$.(2) *The map*

$$\psi : R(\mathbf{X}) \otimes_{R(\mathbf{X})^G} R(X)^G \rightarrow R(X), \quad \mathbf{s} \otimes s \mapsto \varphi^*(\mathbf{s})s,$$

is an isomorphism.(3) *The multiplication map induces an isomorphism*

$$k[C] \otimes R(X)^G \cong R(X).$$

Moreover, the G -module $R(X)^G$ is generated by the monomials in s_1, \dots, s_r and the s_D , $D \in \mathcal{D}$.

Proof. (1) Recall that $\operatorname{div}_{\mathbf{X}}(\mathbf{f}_i) = \mathbf{X}_i - \sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle \mathbf{D}$ with obvious notation. Thus,

$$\operatorname{div}_X(\varphi^*\mathbf{f}_i) = \varphi^*(\mathbf{X}_i) - \sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle D.$$

It follows that $\varphi^*[\mathbf{X}_i] = \sum_{j=1}^n v_j(\mathbf{f}_i)[X_j]^G$, which implies our assertion.

(2) It suffices to show that the restriction

$$\psi^U : R(\mathbf{X})^U \otimes_{R(\mathbf{X})^G} R(X)^G \rightarrow R(X)^U$$

is an isomorphism. The left-hand side is generated as a $k[C]$ -algebra by the $\mathbf{s}_D \otimes 1$, $D \in \mathcal{D}$, and the $1 \otimes s_i$, $i = 1, \dots, n$. Moreover, $\psi(\mathbf{s}_D \otimes 1) = s_D$ by (4.3.2), and $\psi(1 \otimes s_i) = s_i$. So the assertion follows from Proposition 4.2.3.

(3) follows from (2) together with the isomorphism $k[C] \otimes R(X)^G \cong R(X)^G$ given by the multiplication map. \square

In geometric terms, the quotient map $\tilde{X} \rightarrow \tilde{X} // G$ is the pull-back of the corresponding map

$$\mathbf{q} : \tilde{\mathbf{X}} \rightarrow \tilde{\mathbf{X}} // G \cong \mathbb{A}^r$$

under the natural morphism

$$\tilde{\varphi} // G : \tilde{X} // G \rightarrow \tilde{\mathbf{X}} // G.$$

Moreover, $\tilde{X} // \mathbf{G} \cong C \times \tilde{X} // G$ and $\tilde{\varphi} // \mathbf{G}$ factors through the second projection $\tilde{X} // \mathbf{G} \rightarrow \tilde{X} // G$. It follows that the quotient morphism

$$q: \tilde{X} \rightarrow \tilde{X} // G \cong \mathbb{A}^n$$

is flat, and its fibers are exactly the products of C with the fibers of \mathbf{q} .

Also, \tilde{X} is a trivial principal C -bundle over $\tilde{X}/C := \operatorname{Spec} R(X)^C$. We may regard $R(X)^C$ as the total coordinate ring of X with respect to the canonical linearizations defined in 4.1, so that the principal bundle $\tilde{X} \rightarrow \tilde{X}/C$ corresponds to the variation of linearizations.

4.4. The automorphism group of a complete toroidal variety

We shall extend part of the results of 2.4 to the setting of a complete toroidal variety X . We first relate the connected automorphism group of X (a linear algebraic group) to that of a standard embedding, as in 4.1. Let $\varphi: X \rightarrow \mathbf{X}$ be the morphism defined in 4.3 and consider its Stein factorization

$$X \xrightarrow{\psi} X' \xrightarrow{f} \mathbf{X}.$$

Then $X' = \operatorname{Spec}_{\mathbf{X}}(\varphi_* \mathcal{O}_X)$ is a complete spherical G -variety and ψ, f are equivariant. Let H' be the stabilizer of the base point $\psi(x)$ of X' . Then

$$C \cdot H \subseteq H' \subseteq C \times \mathbf{H} \subseteq N_G(H)$$

and $H'/H = (N_G(H)/H)^0$ is a torus containing C , while $N_G(H')/H' = N_G(H)/H'$ is a finite abelian group. The action of H'/H on G/H extends to an action on X , so that

$$H'/H = \operatorname{Aut}_G^0(X).$$

Since the morphism f is finite, X' is the standard embedding of G/H' . Moreover, $\psi^{-1}(G/H') \cong G \times^{H'} F$, where F is a complete toric variety under H'/H ; note that F is the general fiber of the morphism ψ .

The group $\operatorname{Aut}^0(X)$ acts on X' so that ψ is equivariant (this follows, e.g., by considering the pull-back L of an ample invertible sheaf on X' , and linearizing a positive power of L with respect to $\operatorname{Aut}^0(X)$). This yields a homomorphism of algebraic groups

$$\psi_*: \operatorname{Aut}^0(X) \rightarrow \operatorname{Aut}^0(X')$$

with kernel a closed subgroup of $\operatorname{Aut}(F)$ containing H'/H . The structure of $\operatorname{Aut}(F)$ is well known; in particular, H'/H is a maximal torus of this group, and hence of the kernel of ψ_* . (However, there may exist no homomorphism $\varphi_*: \operatorname{Aut}^0(X) \rightarrow \operatorname{Aut}^0(\mathbf{X})$, e.g., when $G = \operatorname{Spin}_{n+1}$ where $n \geq 3$, and $X \subset \mathbb{P}^{n+1}$ is a non-singular quadric hypersurface. Then $X' = X$ and $\mathbf{X} = \mathbb{P}^n$, so that $\operatorname{Aut}^0(X) \cong \operatorname{PSO}_{n+2}$ while $\operatorname{Aut}^0(\mathbf{X}) \cong \operatorname{PSL}_{n+1}$.)

We may now state our generalization of Theorem 2.4.2(2):

Theorem 4.4.1.

- (1) *The closed connected subgroup $\text{Aut}^0(X, \partial X) \subseteq \text{Aut}^0(X)$ is independent of the complete toroidal embedding X of G/H . The Lie algebra of this group is $\Gamma(X, \mathcal{S}_X)$, where \mathcal{S}_X denotes the direct image of the logarithmic tangent sheaf of $X^{\leq 1}$. We have a split exact sequence of Lie algebras*

$$0 \rightarrow \mathfrak{h}'/\mathfrak{h} \rightarrow \Gamma(X, \mathcal{S}_X) \xrightarrow{\psi_*} \Gamma(X', \mathcal{S}_{X'}) \rightarrow 0 \quad (4.4.1)$$

and an isomorphism of Lie algebras

$$\Gamma(X', \mathcal{S}_{X'}) \cong \Gamma(\mathbf{X}, \mathcal{S}_{\mathbf{X}}). \quad (4.4.2)$$

In particular, the group $\text{Aut}^0(X, \partial X)$ is reductive, with connected center H'/H .

- (2) *Let $G' \subseteq \text{Aut}^0(X)$ be a closed connected subgroup containing the image of G , and centralizing H'/H . Then G' is reductive, its connected center is contained in H'/H , and the G' -variety X is toroidal, with the same colors as those of the G -variety X .*

Proof. (1) We may choose a desingularization

$$\pi : Y \rightarrow X$$

which is equivariant with respect to $\text{Aut}^0(X)$. Then Y is a complete toroidal embedding of G/H , and we have an injective homomorphism $\text{Aut}^0(Y) \rightarrow \text{Aut}^0(X)$ which restricts to

$$\pi_* : \text{Aut}^0(Y, \partial Y) \rightarrow \text{Aut}^0(X, \partial X).$$

On the other hand, $\text{Aut}^0(X, \partial X)$ lifts to a subgroup of $\text{Aut}^0(Y)$, which preserves the open G -orbit and hence ∂Y . Thus, π_* is an isomorphism.

Let \mathcal{S}_Y be the sheaf of vector fields on Y which preserve the boundary. Then the sheaf $\pi_*(\mathcal{S}_Y)$ is equipped with a map to \mathcal{S}_X . Using the local structure of toroidal varieties as in the proof of [6, Proposition 2.7.1], one checks that this map is an isomorphism of locally free sheaves (indeed, one reduces to the toric case, where $\pi_*(\mathcal{S}_Y) = \mathcal{S}_X = \mathcal{O}_X \otimes \mathfrak{g}$). Moreover, the Lie algebra of $\text{Aut}^0(Y, \partial Y)$ is $\Gamma(Y, \mathcal{S}_Y)$ and does not depend on the complete toroidal non-singular embedding Y of G/H ; see, e.g., [6, Section 2]. Thus, the Lie algebra $\Gamma(X, \mathcal{S}_X) \cong \Gamma(Y, \mathcal{S}_Y)$ depends only on G/H , so that the same holds for $\text{Aut}^0(X, \partial X)$ (regarded as a group of automorphisms of G/H). In fact, the G -variety X is pseudo-free in the sense of [23, Section 2], and the sheaf \mathcal{S}_X is the same as the one defined there.

Likewise, the local structure of toroidal varieties yields an exact sequence of locally free sheaves

$$0 \rightarrow \mathcal{O}_X \otimes (\mathfrak{h}'/\mathfrak{h}) \rightarrow \mathcal{S}_X \rightarrow \psi^*(\mathcal{S}_{X'}) \rightarrow 0.$$

Taking global sections yields the exact sequence (4.4.1), since

$$\psi_*(\mathcal{O}_X) = \mathcal{O}_{X'} \quad \text{and} \quad H^1(X, \mathcal{O}_X) = 0.$$

On the other hand, the isomorphism (4.4.2) follows from [23, Corollary 6.2]. Together with Theorem 2.4.2, this implies that the Lie algebra $\Gamma(X', \mathcal{S}_{X'})$ is semi-simple. In turn, this implies that $\text{Aut}^0(X, \partial X)$ is reductive with connected center H'/H (this may also be proved directly, along the lines of Lemma 2.4.3).

(2) The kernel of the restriction of ψ_* to G' is a subgroup of $\text{Aut}(F)$ centralized by H'/H , i.e., a subgroup of H'/H . Thus, it suffices to show that $\psi_*(G') \subseteq \text{Aut}^0(X')$ is semi-simple, and the $\psi_*(G')$ -variety X' is toroidal. In other words, we may assume that X is a standard embedding. Recall that $f: X \rightarrow \mathbf{X}$ induces a bijection between the boundary divisors of X and those of \mathbf{X} , which preserves the fixed divisors

We claim that a boundary divisor X_i is either G' -stable or contains no G' -orbit. Indeed, if X_i is moved by G' , then it is not fixed, and hence semi-ample. This yields a surjective morphism $\varphi: X \rightarrow X'$ with connected fibers, where X' is a wonderful variety of rank 1 with ample boundary divisor X'_i , and $X'_i = \varphi^{-1}(X_i)$ (as sets). The group G' acts on X' so that φ is equivariant, and moves X'_i . Hence G' acts transitively on X' , which implies our claim.

Using that claim, we may argue as in the proof of Theorem 2.4.2: let I be the set of indices of G' -stable boundary divisors (so that I contains the indices of fixed divisors) and let $X_I := \bigcap_{i \in I} X_i$. Then X_I is G' -stable, and any proper G -stable subvariety contains no G' -orbit. Thus, X_I is a unique orbit of G' , and hence of a Levi subgroup $G'' \subseteq G'$ containing G . By the claim again, the G'' -variety X is toroidal with boundary divisors X_i , $i \in I$. Moreover, $G'' \subseteq G' \subseteq \text{Aut}^0(X, \partial_I X)$. It follows that G' is semi-simple, by the first step together with Lemma 2.4.3.

The assertion on colors is also checked by adapting the proof of Theorem 2.4.2. \square

Corollary 4.4.2. *Let X be a complete spherical G -variety and denote by $\text{Aut}_{\text{cr}}(X)$ the largest closed connected subgroup of $\text{Aut}^0(X)$ which stabilizes all the G -orbits. Then $\text{Aut}_{\text{cr}}(X)$ is reductive and depends only on the open G -orbit in X .*

If X is toroidal, then $\text{Aut}_{\text{cr}}(X) = \text{Aut}^0(X, \partial X)$.

Proof. Consider the rational G -equivariant map $f: X \dashrightarrow \mathbf{X}$ as in 4.3. By [32], there exists a sequence of blowups $\pi: Y \rightarrow X$ with non-singular G -stable centers, such that the composition $f \circ \pi: Y \rightarrow \mathbf{X}$ is regular. Thus, Y is a complete toroidal G -variety, and the action of $\text{Aut}_{\text{cr}}(X)$ on X lifts to an action on Y . Clearly, the latter action preserves the boundary ∂Y , and hence we may regard $\text{Aut}_{\text{cr}}(X)$ as a subgroup of $\text{Aut}^0(Y, \partial Y)$.

On the other hand, $\text{Aut}^0(Y, \partial Y)$ preserves all the G -orbits in Y , since their closures are exactly the partial intersections of boundary divisors. So, via the homomorphism $\text{Aut}^0(Y) \rightarrow \text{Aut}^0(X)$ induced by π , the subgroup $\text{Aut}^0(Y, \partial Y)$ is mapped onto $\text{Aut}_{\text{cr}}(X)$. We have shown that $\text{Aut}_{\text{cr}}(X) = \text{Aut}^0(Y, \partial Y)$. By Theorem 4.4.1, this group is reductive and depends only on the open G -orbit. \square

Via this corollary, we may associate with any spherical homogeneous space G/H a connected reductive group $\text{Aut}_{\text{cr}}(G/H)$ that we call the *group of completely regular automorphisms*, by analogy with the completely regular differential operators of [23].

It would be interesting to determine the spherical homogeneous spaces G/H whose group of completely regular automorphisms strictly contains the image of G . By Theorem 4.4.1, this reduces to determining the wonderful varieties X such that $\text{Aut}^0(X, \partial X)$ strictly contains the image of G . Those of rank 0 are exactly the flag varieties G/P which are homogeneous under a larger connected semi-simple group; their classification is well known, see [16]. Examples of

rank 1 include the quadrics of dimension 6 (where $G = G_2$ and $\text{Aut}^0(X, \partial X) = \text{SO}_7$) and of dimension 7 (where $G = \text{Spin}_7$ and $\text{Aut}^0(X, \partial X) = \text{PSO}_8$).

Acknowledgments

A large part of this work was written in January 2006, during a staying at the University of Georgia. I would like to thank this institution for its support; also, many thanks are due to Valery Alexeev and Bill Graham for hospitality and fruitful discussions. Finally, I thank the referee for his careful reading and valuable comments.

References

- [1] A. A'Campo–Neuen, J. Hausen, S. Schröer, Homogeneous coordinates and quotient presentations for toric varieties, *Math. Nachr.* 246/247 (2002) 5–19.
- [2] D. Akhiezer, Equivariant completion of homogeneous algebraic varieties by homogeneous divisors, *Ann. Glob. Anal. Geom.* 1 (1983) 49–78.
- [3] V. Alexeev, M. Brion, Stable reductive varieties I: Affine varieties, *Invent. Math.* 157 (2004) 227–274.
- [4] V. Alexeev, M. Brion, Moduli of affine schemes with reductive group action, *J. Algebraic Geom.* 14 (2005) 83–117.
- [5] F. Berchtold, J. Hausen, Homogeneous coordinate rings for algebraic varieties, *J. Algebra* 266 (2003) 636–670.
- [6] F. Bien, M. Brion, Automorphisms and local rigidity of regular varieties, *Compos. Math.* 104 (1996) 1–26.
- [7] P. Bravi, G. Pezzini, Wonderful varieties of type D , *Represent. Theory* 9 (2005) 578–637.
- [8] M. Brion, Représentations exceptionnelles des groupes semi-simples, *Ann. Sci. École Norm. Sup. (4)* 18 (1985) 345–387.
- [9] M. Brion, On spherical varieties of rank one (after D. Ahiezer, A. Huckleberry, D. Snow), in: *Group Actions and Invariant Theory*, Montreal, PQ, 1988, in: CMS Conf. Proc., vol. 10, Amer. Math. Soc., Providence, RI, 1989, pp. 31–41.
- [10] M. Brion, Groupe de Picard et nombres caractéristiques des variétés sphériques, *Duke Math. J.* 58 (1989) 397–424.
- [11] R. Brylinski, B. Kostant, Nilpotent orbits, normality, and Hamiltonian group actions, *J. Amer. Math. Soc.* 7 (1994) 269–298.
- [12] R. Chirivì, P. Littelmann, A. Maffei, Equations defining symmetric varieties and affine Grassmannians, preprint, 2006.
- [13] R. Chirivì, A. Maffei, The ring of sections of a complete symmetric variety, *J. Algebra* 261 (2003) 310–326.
- [14] D. Cox, The homogeneous coordinate ring of a toric variety, *J. Algebraic Geom.* 4 (1995) 17–50.
- [15] C. De Concini, C. Procesi, Complete symmetric varieties, in: *Invariant Theory*, Montecatini, 1982, in: *Lecture Notes in Math.*, vol. 996, Springer, Berlin, 1983, pp. 1–44.
- [16] M. Demazure, Automorphismes et déformations des variétés de Borel, *Invent. Math.* 39 (1977) 179–186.
- [17] D. Edidin, W. Graham, Equivariant intersection theory, *Invent. Math.* 131 (1998) 595–634.
- [18] E. Javier Elizondo, K. Kurano, K.-I. Watanabe, The total coordinate ring of a normal projective variety, *J. Algebra* 276 (2004) 625–637.
- [19] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math., vol. 52, Springer, New York, 1977.
- [20] A. Huckleberry, D. Snow, Almost-homogeneous Kähler manifolds with hypersurface orbits, *Osaka J. Math.* 19 (1982) 763–786.
- [21] F. Knop, The Luna–Vust theory of spherical embeddings, in: *Proceedings of the Hyderabad Conference on Algebraic Groups*, Manoj Prakashan, Madras, 1991, pp. 225–250.
- [22] F. Knop, Über Hilberts vierzehntes Problem für Varietäten mit Kompliziertheit eins, *Math. Z.* 213 (1993) 33–35.
- [23] F. Knop, A Harish-Chandra homomorphism for reductive group actions, *Ann. of Math. (2)* 140 (1994) 253–288.
- [24] F. Knop, Automorphisms, root systems, and compactifications of homogeneous varieties, *J. Amer. Math. Soc.* 9 (1996) 153–174.
- [25] F. Knop, H. Kraft, D. Luna, T. Vust, Local properties of algebraic group actions, in: *Algebraic Transformation Groups and Invariant Theory*, Birkhäuser, Basel, 1989, pp. 63–75.
- [26] D. Luna, Toute variété magnifique est sphérique, *Transform. Groups* 1 (1996) 245–258.
- [27] D. Luna, Grosses cellules pour les variétés sphériques, in: *Algebraic Groups and Lie Groups*, in: *Austral. Math. Soc. Lect. Ser.*, vol. 9, Cambridge Univ. Press, 1997, pp. 267–280.
- [28] D. Luna, Variétés sphériques de type A , *Publ. Math. Inst. Hautes Études Sci.* 94 (2001) 161–226.

- [29] V.L. Popov, Contraction of the actions of reductive algebraic groups, *Math. USSR Sb.* 58 (1987) 311–335.
- [30] V.L. Popov, E.B. Vinberg, *Invariant Theory*, *Enclycopaedia Math. Sci.*, vol. 55, Springer, New York, 1994.
- [31] C.P. Ramanujam, A note on automorphism groups of algebraic varieties, *Math. Ann.* 156 (1964) 25–33.
- [32] Z. Reichstein, B. Youssin, Equivariant resolution of points of indeterminacy, *Proc. Amer. Math. Soc.* 130 (2002) 2183–2187.
- [33] L.E. Renner, The cone of semi-simple monoids with the same factorial hull, *arXiv: math.AG/0603222*.
- [34] R.W. Richardson, A rigidity theorem for subalgebras of Lie and associative algebras, *Illinois J. Math.* 11 (1967) 92–110.
- [35] R.W. Richardson, On orbits of algebraic groups and Lie groups, *Bull. Austral. Math. Soc.* 25 (1) (1982) 1–28.
- [36] A. Rittatore, *Monoïdes algébriques et plongements des groupes*, PhD thesis, Grenoble, 1997, available at <http://www-fourier.ujf-grenoble.fr/THESE/ps/trittatore97.ps.gz>.
- [37] A. Rittatore, Very flat reductive monoids, *Publ. Mat. Urug.* 9 (2002) 93–121.
- [38] E.B. Vinberg, The asymptotic semigroup of a semi-simple Lie group, in: *Semigroups in Algebra, Geometry and Analysis*, Oberwolfach, 1993, in: *de Gruyter Exp. Math.*, vol. 20, de Gruyter, Berlin, 1995, pp. 293–310.
- [39] E.B. Vinberg, On reductive algebraic semigroups, in: *Lie Groups and Lie Algebras*, E.B. Dynkin’s Seminar, in: *Amer. Math. Soc. Transl. Ser. 2*, vol. 169, Amer. Math. Soc., Providence, RI, 1995, pp. 145–182.
- [40] B. Wasserman, Wonderful varieties of rank two, *Transform. Groups* 1 (1996) 375–403.